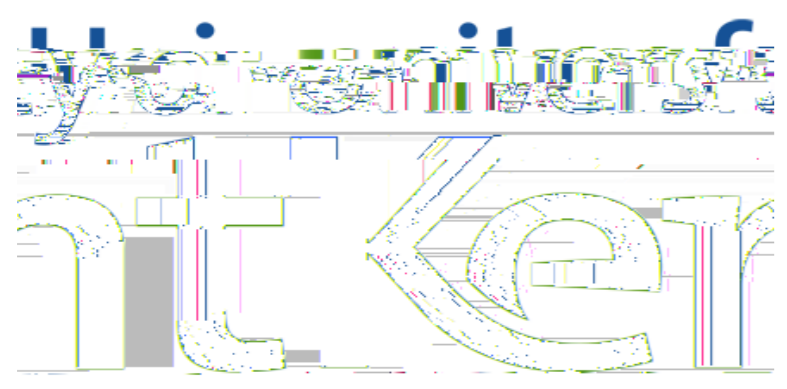


Non-Hermitian ensembles and Painlevé critical asymptotics



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Normal matrix model

We are interested in the normal random matrix model defined by

$$dP_N(z_1, z_2, \dots, z_N; t) = \frac{1}{Z_N(t)^N} \prod_{j=1}^N (z_j)^2 e^{-NV_t^{(s)}(z_j)} dA(z_j);$$

with $z_j \in \mathbb{C}$ and potential

$$V_t^{(s)}(z) = jz^{2s} - t(z^s + \bar{z}^s); \quad s \in \mathbb{N};$$

The eigenvalues z_1, \dots, z_N display an interesting behaviour:

Figure 1: The limiting eigenvalue distribution is supported on the interior of the orange curves. Here $s = 11$ and $t = t_c - 0.1$ (left), $t = t_c$ (centre) and $t = t_c + 0.1$ (right). At the special value $t = t_c$, the support becomes disconnected.

In this poster our goal is to investigate the partition function $Z_N(t)$ near the critical value $t = t_c = 1 = \frac{1}{2s}$.

Reduction to the Ginibre ensemble

The first observation is that we can use symmetry to write $Z_N(t)$ as an average over the Ginibre ensemble:

$$Z_{Ns}(t) = c_{N;s} \int_{\mathbb{C}^N} \prod_{j=1}^N Z_N^{(j)}(x); \quad j := 2, 1, \frac{l+1}{s};$$

where

$$Z_N^{(j)}(x) = \int_{\mathbb{C}^N} \prod_{j=1}^N (z_j)^2 e^{-Njz_j^2} e^{-Njz_j^2} dA(z_j); \quad x := t^{\frac{1}{2s}};$$

Criticality now corresponds to the spectral variable x colliding with the boundary of the circular law (i.e. $|x| = 1$). When $|x| < 1$ (sub-critical), the asymptotics were obtained in [2].

Painlevé and non-Hermitian matrix integrals

Our main result for finite N characterizes the partition function as a solution of the σ -form of Painlevé V.

Theorem 1. The 'reduced' partition functions $Z_N^{(j)}(x)$ can be written as

1. An average over the CUE:

$$Z_N^{(j)}(x) = c_N \int_{\mathbb{C}^N} \prod_{j=1}^N e^{-\frac{i}{4}j|1 + e^i j \bar{z}_j|^2} e^{-N|x|^2 e^i j} dA(z_j);$$

2. The σ -form of Painlevé V:

$$Z_N^{(j)}(x) = c_N \exp \int_0^x \frac{y_N(t) + \frac{N}{2}}{t} dt$$

where $y_N(t)$ satisfies the equation

$$(t^{\theta})^2 \left[t^{\theta} + 2(t^{\theta})^2 + (N - \theta)^2 \right]^2 + 4(t^{\theta} - \frac{\theta}{2})^2 (t^{\theta} + N) = 0; \quad (1)$$

with initial condition

$$y_N(t) = \frac{N}{2} + \frac{t}{2}; \quad t \rightarrow 0;$$

The first part above can be arrived at by a judicious inspection of formulas in [2]. Then the second part is a consequence of the first due to results of Forrester and Witte '02.

Large N asymptotic results

Asymptotic results for related orthogonal polynomials have been studied in various works, but the critical case only very recently in [1]. For the partition function, we obtain:

Theorem 2. If $s = 2k$, where $k \in \mathbb{N}$, then for

$$|x| < 1 - \frac{u}{N}; \quad u \in \mathbb{R}$$

we have the following asymptotics:

$$\frac{Z_N^{(2k)}(x)}{E_{N;k}} = \exp \left(\int_u^x \frac{v(z)}{z} dz \right) (1 + o(1)); \quad N \rightarrow \infty;$$

uniformly for u in compact subsets of \mathbb{R} , where $E_{N;k}$ is a completely explicit pre-factor. The function v satisfies the σ -form of the Painlevé IV equation:

$$(v^{\theta})^2 + 4(v^{\theta})^2(v^{\theta} + k) - (sv^{\theta} - v^{\theta})^2 = 0; \quad (2)$$

subject to the boundary condition

$$v(s) = ks - \frac{k}{s} + O(s^{-3}); \quad s \rightarrow \infty;$$

We believe this result persists to non-integer k , indeed a naive rescaling of equation (1) reproduces exactly the Painlevé IV in (2). The advantage of integer k is the duality (Forrester and Rains '08):

$$\frac{Z_N^{(2k)}(x)}{E_{N;k}} = \int_{[0;1]^k} \prod_{j=1}^k e^{-Njx^2 r_j} \left(1 + \frac{r_j}{N} \right)^N \prod_{j=1}^k (r_j)^2 dr_j; \quad (3)$$

making $N \rightarrow \infty$ asymptotics easy to compute. For k not integer, we use Riemann–Hilbert techniques.