# Non-Hermitian ensembles and Painlevé critical asymptotics


**Alfredo Deaño & Nick Simm**

University of Kent, University of Sussex

A.Deano-Cabrera@kent.ac.uk, N.J.Simm@sussex.ac.uk


## Normal matrix model

We are interested in the normal random matrix model defined by

$$dP_N(z_1, z_2, \ldots, z_N; t) = \frac{1}{Z_N(t)} |\ (z)|^2 \prod_{j=1}^{N} e^{-NV_t^{(s)}(z_j)} dA(z_j),$$

with $z_j \in \mathbb{C}$ and potential

$$V_t^{(s)}(z) = |z|^{2s} - t(z^s + \bar{z}^s), \qquad s \in \mathbb{N}.$$

The eigenvalues $z_1, \ldots, z_N$ display an interesting behaviour:

**Figure 1:** The limiting eigenvalue distribution is supported on the interior of the orange curves. Here $s = 11$ and $t = t_c - 0.1$ (left), $t = t_c$ (centre) and $t = t_c + 0.1$ (right). At the special value $t = t_c$, the support becomes disconnected.

In this poster our goal is to investigate the partition function $Z_N(t)$ near the critical value $t = t_c = 1/\sqrt{s}$.

## Reduction to the Ginibre ensemble

The first observation is that we can use symmetry to write $Z_N(t)$ as an average over the Ginibre ensemble:

$$Z_{Ns}(t) = C_{N,s} \prod_{l=0}^{s-1} Z_N^{(\ _l)}(x), \qquad _l := 2\left(1 - \frac{l+1}{s}\right),$$

where

$$Z_N^{(\ )}(x) = \int_{\mathbb{C}^N} |\ (z)|^2 \prod_{j=1}^{N} |z_j - x|\ e^{-N|z_j|^2} dA(z_j), \qquad x := t\sqrt{s}.$$

Criticality now corresponds to the spectral variable $x$ colliding with the boundary of the circular law (*i.e.* $|x| = 1$). When $|x| < 1$ (sub-critical), the asymptotics were obtained in [2].

## Painlevé and non-Hermitian matrix integrals

Our main result for finite $N$ characterizes the partition function as a solution of the -form of Painlevé V.

*Theorem 1. The 'reduced' partition functions* $Z_N^{(\ )}(x)$ *can be written as*

1. *An average over the CUE:*

$$Z_N^{(\ )}(x) = c_{N,\ } \left\langle \prod_{j=1}^{N} e^{-\frac{i\ \ _j}{4}} |1 + e^{i\ _j}|^{\frac{}{2}} e^{Nx^2 e^{i\ _j}} \right\rangle_{\mathrm{CUE}}.$$

2. *The -form of Painlevé V:*

$$Z_N^{(\ )}(x) = c_{N,\ } \exp\left( \int_0^{Nx^2} \frac{y_N(t) + \frac{N}{2}}{t}\, dt \right)$$

*where* $y_N(t)$ $(t)$ *satisfies the equation*

$$(t\ '')^2 - [\ -t\ ' + 2(\ ')^2 + (N - \ )\ ']^2 + 4\ '(\ ' - \tfrac{}{2})^2(\ ' + N) = 0, \quad (1)$$

*with initial condition*

$$(t) \sim -\frac{N}{2} + \frac{t}{2}, \qquad t \to 0.$$

The first part above can be arrived at by a judicious inspection of formulas in [2]. Then the second part is a consequence of the first due to results of Forrester and Witte '02.

## Large $N$ asymptotic results

Asymptotic results for related orthogonal polynomials have been studied in various works, but the critical case only very recently in [1]. For the partition function, we obtain:

*Theorem 2. If* $= 2k$, *where* $k \in \mathbb{N}$, *then for*

$$|x|^2 = 1 - \frac{u}{\sqrt{N}}, \qquad u \in \mathbb{R}$$

*we have the following asymptotics:*

$$\frac{Z_N^{(2k)}(x)}{E_{N,k}} = \exp\left( -\sqrt{N} ku - \int_u^{\infty} v(\ )\, d\ \right)(1 + o(1)), \quad N \to \infty,$$

*uniformly for* $u$ *in compact subsets of* $\mathbb{R}$, *where* $E_{N,k}$ *is a completely explicit pre-factor. The function* $v$ *satisfies the -form of the Painlevé IV equation:*

$$(v'')^2 + 4(v')^2(v' + k) - (sv' - v)^2 = 0, \qquad (2)$$

*subject to the boundary condition*

$$v(s) = -ks - \frac{k}{s} + O(s^{-3}), \qquad s \to \infty.$$

We believe this result persists to non-integer $k$, indeed a naive rescaling of equation (1) reproduces exactly the Painlevé IV in (2). The advantage of integer $k$ is the *duality* (Forrester and Rains '08):

$$\frac{Z_N^{(2k)}(x)}{\tilde{E}_{N,k}} = |x|^{2Nk + 2k^2} \int_{[0,\infty)^k} \prod_{j=1}^{k} e^{-\sqrt{N}|x|^2 r_j} \left(1 + \frac{r_j}{\sqrt{N}}\right)^N |\ (r)|^2 dr, \quad (3)$$

making $N \to \infty$ asymptotics easy to compute. For $k$ not integer, we use Riemann–Hilbert techniques.