Extreme Value Statistics of Maximal Eigenvalues of Random Matrices

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PRL 97 160201 (2006)

Plan of Talk

- Motivation - statistics and string theory
- Coulomb gas formulation for Gaussian ensembles
- Solution of problem by hard wall constraint- half Hilbert transfrom
- Some numerical tests

Conditioned matrix ensembles

M -random matrix e.g. from GOE, GUE or GSE ensemble

Say M describes Hamiltonian of vibrations about some stationary point in the energy landscape or the stability matrix of a fixed point in a dynamical system. Symmetry and complexity determine Gaussian statistics but in an experiment we can only observe systems where all eigenvalues are > z

(i) What is the probability of finding such a matrix ?
(ii) What are its spectral properties ?

Anthropic principles

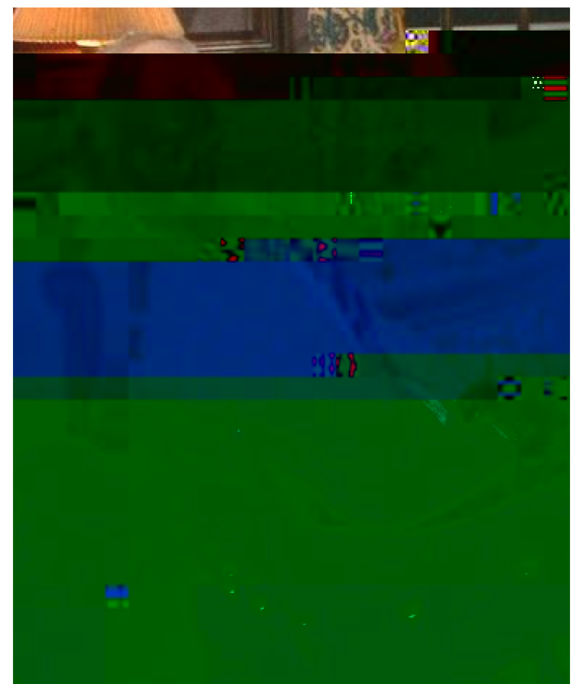
The current form of string theory is believed to have 10^{500} possible vacua -the Landscape-each corresponding to a possible Universe and physical constants.

It seems unlikely that life can evolve in a generic universe so we are lucky to be here - anthropic principle this is the only type of Universe we could prossibly see, there could be lots of others (statistically dominating) which are quite different to ours.

What properties of our universe depend of the details of string theory and which depend simply on the statistics of the complex/random landscape ?

Anthropic Reasoning

Fred Hoyle- a ^{12}C based life form reasoned there should be a mechanism that permits sufficient ^{12}C for life to exist,he then found the ^{4}He+^{8}Be resonnance of ^{12}C that is responsible



Stephen Weinberg: the cosmological constant is small but if one takes the maximal value avoiding a « Big Rip » this gives a surprisngly good estimate

Estimating the probability of a stable universe

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Coulomb Gas Formulation

$$P(\{\lambda_i\}) = B_N \exp\ -\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i^2 \quad {}_{i<} \quad |\,\lambda_i$$

Hard Wall Constraint

P_N

Density functional method

$$f(\mu) = \frac{1}{N}\sum_{i=1}^{N}\delta(\mu-\mu_i)$$

Normalised density field

$$H[f] = -N^2\Sigma[f]$$

Superextensive energy scaling

$$\Sigma[f] = -\frac{1}{2}\int f(\mu)\mu^2 d\mu + \frac{1}{2}\int f(\mu) f(\mu')\ln(|\mu-\mu'|)d\mu d\mu'$$

Constraints on f: $f(\mu)=0 \;\; for \;\; \mu<z \quad ; \quad \int f(\mu)d\mu = 1$

$$Z_N(z) = \int d[f]J[f]\exp(N^2\beta\Sigma[f])$$

Jacobian to pass from coordinates
To density field

Functional Integral

$$J[f] = C_N \int \prod_i d\mu_i \delta(Nf(\mu) - \sum_i \delta(\mu - \mu_i))$$

$$= C_N \int d[g] \exp(N \int f(\mu) g(\mu) + N \ln\left(\int \exp(-g(\mu)) d\mu\right))$$

Jacobian term is exp(O(N)) and is thus negligible with respect to the energy term exp(O(N^2)).

Formally $$J[f] = C_N^{''} \exp\left(-N \int d\mu f(\mu) \ln(f(\mu))\right)$$

If large N saddle-point can be justified above - mean field like entropy term (slightly more to it than that).

Leading order behaviour

$$Z_N(z) = D_N \exp(N^2 \beta S(z) + O(N))$$

$$S[z] = \max_f \{\Sigma[f]\}$$

Saddle point equation,
gives thermodynamically

$$\mu = P \int_z^{\infty} d\mu' \frac{f(\mu)}{\mu - \mu'}$$

$$\mu = z + x \Rightarrow z + x = (H_+ f)(x)$$

$$H_+ F = P \int_0^{\infty} dx' \frac{F(x')}{x - x'}$$

Recovering Wigner's Semi-Circle Law

However f must be real and positive, but f(0)<0 if $L(z) < -2z$

$$\Rightarrow z < -\sqrt{2}$$

$$f(\mu) = \frac{1}{\pi}\sqrt{2-\mu^2}$$

Wigner semi-circle law when z=-√2- but this solution is Ok for all z<-√2 as it respects the boundary conditions

Setting the boundary to the left of z=-√2 does not affect the Coulomb gas - the wall does not touch the Wigner semi-cirlcle

$$\Rightarrow S(z) = S(-\sqrt{2}) = S(-\infty) \;\; for\; z < -\sqrt{2}$$

Modified density function

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z=1/2

z=-1

z=0

Effect of moving the barrier position z on the eigenvalue density distribution

Comparsion with Numerics

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Analytic formula for
f for z=0 compared
to that found by
numerical diagonalizing
of 6x6 matrices

Computing S(z)

$$\frac{\mu^2}{2} + C = \int_z^{\infty} d\mu' f(\mu') \ln(|\mu - \mu'|) \Rightarrow$$

$$\int d\mu f(\mu) \frac{\mu^2}{4} + \frac{C}{2} = \frac{1}{2} \int d\mu d\mu' f(\mu) f(\mu') \ln(|\mu - \mu'|)$$

$$\Sigma[f] = \frac{1}{2} \int f(\mu) \mu^2 d\mu - \frac{1}{2} \quad \mu f(\mu) \mu C 4 ($$

Form of S(z)

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$$P_N(\lambda_{\min} > \zeta) = P_N(\mu_{\min} > z) = \frac{Z_N(z)}{Z_N(-\infty)} \quad ; z$$

Matching with Tracy Widom

For

$$P(\lambda_{\min} > t) \approx \exp(-N^2 \beta\theta(\frac{t}{\sqrt{N}} + \sqrt{2}$$

The persistence problem

Probability that all eigenvalues are positive $\approx \exp(-N^2 \beta \theta(0))$

$$\theta(0) = \frac{\ln(3)}{4} \approx 0.274653..$$

Note that $0.25 < 0.274653.. < 0.3291$

Aazami Easther aproximation

Exact result

Aazami Easther numerics

Numerical test of persistence

$$-0.272N^2 - 0.493N + 0.244$$

$\ln(P_N)$

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Direct enumeraton

Can generate the ensembles and do the diagonalization efficiently but for large N you never see matrices with all eigenvalues positive

$$P(\lambda_{\min} > 0) = \frac{m_+}{m}$$

Trick of limited usefulness: if M is a positve matrix then:

$$(v, Mv) > 0 \quad \forall v \Rightarrow (e_i, Me_i) = M_{ii} > 0$$

Conclusions

Coulomb gas formulation of eigenvalues of GOE GUE and GSE eigenvalues allows one to extract probability distribution of smallest eigenvalue far away from its typical value but within the sea

See a modified density of states in the condtioned ensemble quite rich behaviour

Is it possible to calculate the O(N) term and other lower order terms ?

Analysis of non symmetric matrix ensembles, Lyapounov exponents of dynamical fixed points of complex systems.

Are there more efficient ways of doing numerics ?

Can try to look at index distribution of random matrices