



# Arbitrary Unitarily Invariant Random Matrix Ensembles and Supersymmetry

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# Outline

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- the **problem** and its **history**
- if you wish: a little bit about **supersymmetry**
- first step: supersymmetric representation for **norm-dependent** ensembles
- general case: supersymmetric representation for **arbitrary rotation invariant** ensembles
- some results **beyond** orthogonal polynomials

TG, J. Phys. A39 (2006) 12327, J. Phys. A39 (2006) 13191

# The Problem and its History

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Efetov's supersymmetry approach (early 80's) based on **Gaussian assumption** for probability densities.

physics: acceptable because of **local universality**

mathematics: **fundamental restriction** of supersymmetry ?

Hackenbroich, Weidenmüller (1995): universality proof involving supersymmetry and twofold asymptotics, not exact

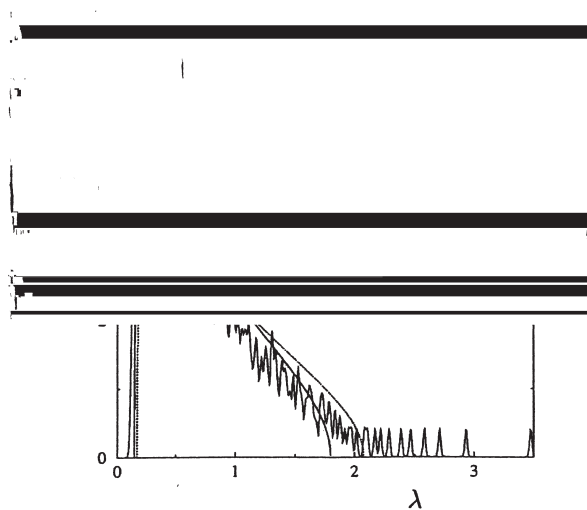
Efetov, Schwiete, Takahashi (2004): superbosonization

TG (2006): algebraic duality, explicit construction

Littelman, Sommers, Zirnbauer (2007): rigorous, threefold way

# Need for Non-Gaussian Probability Densities

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financial correlation matrices

empirical result deviate from  
Gaussian assumption

Laloux, Cizeau, Bouchaud, Potters (1999)

high-energy physics and quantum gravity, probability density:

$$P(H) \sim e^{-p}$$

# Mehta–Mahoux and Factorization

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rotation-invariant probability density:  $P(H) = P(E)$

factorization:  $P(E) = \prod_{n=1}^N P^{(ev)}(E_n)$

$$R_k(E_1, \dots, E_k) = \det_{p,q=1, \dots, k} K_N(E_p, E_q)$$

$$K_N(E_p, E_q) = \sqrt{P^{(ev)}(E_p) P^{(ev)}(E_q)} \sum_{n=0}^{N-1} {}_n E_p \quad {}_n E_q$$

${}_n E_p$  are orthogonal polynomials:



# Supersymmetry — Linear Algebra

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supervectors  $\begin{bmatrix} \mathbf{z} \end{bmatrix}$  and supermatrices  $\begin{bmatrix} \mathbf{a} & \boldsymbol{\mu} \\ & \mathbf{b} \end{bmatrix}$

matrices  $\mathbf{a}, \mathbf{b}$  have **commuting** entries

matrices  $\boldsymbol{\mu},$  have





# Gaussian Integrals over Supervectors

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matrix  $\mathbf{a}$  has commuting entries

$$\int e^{-\mathbf{z}^\dagger \mathbf{a} \mathbf{z}} \mathbf{d} \mathbf{z} = \det^{-1} \mathbf{a} \quad \text{and}$$

# Supersymmetry and Gaussian Random Matrices

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Gaussian ensemble of  $\mathbf{N} \times \mathbf{N}$  Hermitean random matrices  $\mathbf{H}$

$\mathbf{k}$ -point correlations  $\mathbf{R}_k(\mathbf{x}_1, \dots, \mathbf{x}_k) = \frac{1}{\prod_{p=1}^k \mathcal{J}_p} \mathbf{Z}_k(\mathbf{x}, \perp \mathbf{J}) \Big|_{J=0}$

generating function obeys the identity (yes, this is exact!)

$$\mathbf{Z}_k(\mathbf{x}, \perp \mathbf{J}) = \int d\mathbf{H} e^{-\text{tr} \mathbf{H}^2} \prod_{p=1}^k \frac{\det(\mathbf{H} - \mathbf{x}_p - \mathbf{J}_p)}{\det(\mathbf{H} - \mathbf{x}_p \perp \mathbf{J}_p)}$$

$$= \int d\mathbf{g} e^{-\text{tr} \mathbf{g}^2} \det \mathbf{g}^{-N} \det(\mathbf{g} - \mathbf{x} - \mathbf{J})$$

where  $\mathbf{g}$  is a  $\mathbf{k} \times \mathbf{k}$  supermatrix

→ drastic reduction of dimensions

# Posing the Problem as a Structural Issue

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can we generalize this to non-Gaussian probability densities ?

is there an identity of the form

$$\int d\mathbf{H} \mathbf{P}(\mathbf{H}) \prod_{p=1}^k \frac{\det(\mathbf{H} - \mathbf{x}_p - \mathbf{J}_p)}{\det(\mathbf{H} - \mathbf{x}_p - \mathbf{J}_p^\perp)} = \int d\mathbf{Q} \det \mathbf{g}^{-N}(\mathbf{Q}) \delta(\mathbf{Q} - \mathbf{x} - \mathbf{J})$$

given an arbitrary rotation-invariant  $\mathbf{P}(\mathbf{H})$ , what is  $\mathbf{Q}$  ?

# First Step: Norm–dependent Ensembles

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consider

# Examples

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always  $\mathbf{u} = \text{tr} \mathbf{H}^2$  and  $\mathbf{w} = \text{trg}^2$

fixed trace ensemble:

$\mathbf{P}(\cdot)$

# Arbitrary Rotation-invariant Ensembles

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use bosonic fields  $\mathbf{z}_p$  and fermionic fields  $\psi_p$

$$\frac{\det \mathbf{H} - \mathbf{x}_p - \mathbf{J}_p}{\det \mathbf{H} - \mathbf{x}_p - \mathbf{J}_p} \int d\mathbf{z}_p e^{p - i\mathbf{z}_p^\dagger \mathbf{H} - \mathbf{x}_p - \mathbf{J}_p \mathbf{z}_p}$$

$$\int d\psi_p e^{p - i\psi_p^\dagger \mathbf{H} - \mathbf{x}_p - \mathbf{J}_p \psi_p}$$

characteristic function:  $\mathbf{K} \int d\mathbf{H} \mathbf{P} \mathbf{H} e^{p} i \text{tr} \mathbf{H} \mathbf{K}$

Fourier matrix variable:  $\mathbf{K} \sum_{p=1}^k \mathbf{z}_p \mathbf{z}_p^\dagger - \sum_{p=1}^k \psi_p \psi_p^\dagger$

$\mathbf{P} \mathbf{H}$  rotation invariant  $\longrightarrow$   $\mathbf{K}$  rotation invariant

# Duality between Ordinary and Superspace

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introduce  $\mathbf{N} \times \mathbf{k}$  supermatrix  $\mathbf{A} = \begin{bmatrix} \mathbf{z}_1 & \cdots & \mathbf{z}_k \\ 1 & \cdots & k \end{bmatrix}$

$$\mathbf{K} = \sum_{p=1}^k \mathbf{z}_p \mathbf{z}_p^\dagger - \sum_{p=1}^k \begin{matrix} p \\ p \end{matrix} \quad \mathbf{A} \mathbf{A}^\dagger$$

$$\mathbf{B} = \mathbf{A}^\dagger \mathbf{A} = \begin{bmatrix} \mathbf{z}_1^\dagger \mathbf{z}_1 & \cdots & \mathbf{z}_1^\dagger \mathbf{z}_k & \mathbf{z}_1^\dagger 1 & \cdots & \mathbf{z}_1^\dagger k \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{z}_k^\dagger \mathbf{z}_1 & \cdots & \mathbf{z}_k^\dagger \mathbf{z}_k & \mathbf{z}_k^\dagger 1 & \cdots & \mathbf{z}_k^\dagger k \\ - \begin{matrix} 1 \\ 1 \end{matrix} \mathbf{z}_1 & \cdots & - \begin{matrix} 1 \\ 1 \end{matrix} \mathbf{z}_k & - \begin{matrix} 1 \\ 1 \end{matrix} 1 & \cdots & - \begin{matrix} 1 \\ 1 \end{matrix} k \\ \vdots & & \vdots & \vdots & & \vdots \\ - \begin{matrix} k \\ k \end{matrix} \mathbf{z}_1 & \cdots & - \begin{matrix} k \\ k \end{matrix} \mathbf{z}_k & - \begin{matrix} k \\ k \end{matrix} 1 & \cdots & - \begin{matrix} k \\ k \end{matrix} k \end{bmatrix}$$

$\mathbf{K}$  is  $\mathbf{N} \times \mathbf{N}$  ordinary, but  $\mathbf{B}$  is  $\mathbf{k} \times \mathbf{k}$  super

# Equality of Invariants

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for all integers  $m = 1, 2, \dots$  we have the identity

$$\text{tr } \mathbf{K}^m = \text{tr } \mathbf{A}\mathbf{A}^\dagger{}^m = \text{trg } \mathbf{A}^\dagger\mathbf{A}{}^m = \text{trg } \mathbf{B}^m$$

non-trivial connection between ordinary and superspace

remarkable implication for characteristic function

$$\text{tr } \mathbf{K}, \text{tr } \mathbf{K}^2, \text{tr } \mathbf{K}^3, \dots = \text{trg } \mathbf{B}, \text{trg } \mathbf{B}^2, \text{trg } \mathbf{B}^3, \dots$$

same form as function of invariants !!

whole approach will be based on characteristic function



# Spectral Decomposition

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**K** and **B** have the same “relevant” eigenvalues !!

$$\mathbf{K} = \mathbf{V} \mathbf{Y} \mathbf{V}^\dagger$$

# Chain of Equalities

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characteristic function satisfies

**K**      **Y**      **y**      **B**

alternative proof, avoiding the direct use of invariants

# Construction of Generating Function

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# Fourier Superspace Representation

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integrals over fields  $\mathbf{z}_p$  and  $\mathbf{p}$  as usual

$$\mathbf{Z}_k \mathbf{x}^{\perp} \mathbf{J} \int \mathbf{d} \int \mathbf{d} e^{\mathbf{p} - \mathbf{i} \text{trg}} \det \mathbf{g}^{-N} ( - \mathbf{x}^{\perp} - \mathbf{J} )$$

arrive at a Fourier superspace representation only involving the characteristic function

$$\mathbf{Z}_k \mathbf{x}^{\perp} \mathbf{J} \int \mathbf{d} e^{\mathbf{p} - \mathbf{i} \text{trg}} \mathbf{x}^{\perp} \mathbf{J} \mathbf{I}$$

# Generalized Ingham–Siegel–type of integral

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Fourier transform of superdeterminant to power  $-N$

$$I = \int \mathbf{d} \mathbf{g} \, e^{i \text{tr} \mathbf{g}} \, \det \mathbf{g}^{-N} - \prod_{p=1}^k \frac{r_{p1} \, i r_{p1}^N e^{i p} - r_{p1} \frac{N-1}{r_{p2}^{N-1}}}{r_{p2}^{N-1}}$$

almost equal to superdeterminant to power  $-N$

# Probability Density in Superspace

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convolution theorem in superspace yields

$$\mathbf{Z}_k \mathbf{x} \cdot \mathbf{J} = \int \mathbf{d} \mathbf{Q} \det g^{-N} \delta(\mathbf{x} - \mathbf{J})$$

desired probability density is thus Fourier backtransform

$$\mathbf{Q} = \int \mathbf{d} \mathbf{p} e^{i \mathbf{p} \cdot \mathbf{r}(\mathbf{g})}$$

duality between ordinary and superspace connects Fourier transforms !!

# Reduction to Eigenvalue Integrals

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Fourier superspace representation has considerable advantages

$\mathbf{r}$  and  $|\mathbf{r}|$  invariant, apply supersymmetric Harish-Chandra–Itzykson–Zuber integral and do the group integral

$$\mathbf{R}_k \mathbf{x}_1, \dots, \mathbf{x}_k \int d\mathbf{r} \mathbf{B}_k \mathbf{r} e^{p - i \text{trg} \mathbf{x} \mathbf{r}} |\mathbf{r}|$$

with Berezinian (Jacobian)  $\mathbf{B}_k \mathbf{r} = \det \left[ \frac{\mathbf{r}_{p1} - i \mathbf{r}_{q2}}{p, q = 1, \dots, k} \right]$

The full problem is reduced to  $k$  integrals, of which  $k$  can be done trivially. This holds for arbitrary rotation–invariant probability densities  $\mathbf{P} \mathbf{H}$ , including those which do not factorize!

# General Result beyond Orthogonal Polynomials

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another representation for correlation functions

$$R_k(x_1, \dots, x_k) = \int d\mathbf{h} P(\mathbf{h}) R_k^{(\text{fund})}(\mathbf{x} - \mathbf{h})$$

$$\mathbf{h} = (h_{11}, \dots, h_{kk}, i h_{(k+1)(k+1)}, \dots, i h_{(2k)(2k)})$$

convolution of probability density with fundamental

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# Example

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probability density **without** factorization ( $\mathbf{M}_1, \mathbf{M}_2, \dots$ )

$$P(\mathbf{H}) = (\text{tr } \mathbf{H}^{M_1})^{M_2} e^{-p(-\text{tr } \mathbf{H}^2)}$$

correlation functions are linear combinations of determinants

$$R_k(\mathbf{x}_1, \dots, \mathbf{x}_k) = \sum_{\{m\}} \mathbf{a}_{\{m\}} \sum_{\omega} \det \left[ \mathbf{C}_{m_{\omega(p)} m_{\omega(k+q)}}(\mathbf{x}_p, \mathbf{x}_q) \right]_{p,q=1,\dots,k}$$

$$\mathbf{C}_{m_1 m_2}(\mathbf{x}_p, \mathbf{x}_q) = e^{-p(-\mathbf{x}_p^2)} \sum_{n=0}^{N-1} \frac{1}{n} \mathbf{x}_p^{nm_1} \mathbf{x}_q^{nm_2}$$

where  $\mathbf{x}_p^{nm_1}$  and  $\mathbf{x}_q^{nm_2}$  are the components of the vectors  $\mathbf{x}_p$  and  $\mathbf{x}_q$  raised to the powers  $nm_1$  and  $nm_2$  respectively.

# Summary and Conclusions

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- in various applications **non-Gaussian** probability densities
- Mehta–Mahoux theorem needs **factorization**
- first step: **norm-dependent** probability densities
- general case: **arbitrary rotation-invariant** probability densities
- Fourier superspace formulation only builds upon **characteristic function**
- all correlation functions **reduced** to  **$\mathbf{k}$**  (actually  **$\mathbf{k}$** ) integrals
- results **beyond** Mehta–Mahoux theorem
- correlation functions are **convolutions** involving the **fundamental correlations**

work in progress with **M. Kieburg** (Sonderforschungsbereich Transregio 12)