III BRUNEL Workshop on Random Matrix Theory

Matrices, Characters, Quantum (Super)Spin Chains

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Brunel University, Uxbridge, 17/12/07

with P.Vieira, *arXiv:0711.2470*
with A.Sorin and A.Zabrodin, *hep-th/0703147*.

Motivation and Plan

- Classical and quantum integrability are intimately related (not only through classical limit!). Quantization = discretization.

- Quantum spin chain → Discrete *classical* Hirota dynamics for fusion of quantum states (according to representation theory)
  Klumper,Pearce 92', Kuniba,Nakanishi,'92, Krichever,Lupan,Wiegmann, Zabrodin'97

- Based on *Bazhanov-Reshetikhin* (BR) formula for fusion. **Bazhanov,Reshetikhin'90**
  Direct proof of BR formula seems to be absent (but see **Cherednik'88**)
  We fill this gap using the gl(K|M) character technique. **V.K., Vieira'07**

- Solution of Hirota eq. for (super)spin chain in terms of Baxter TQ-relation.

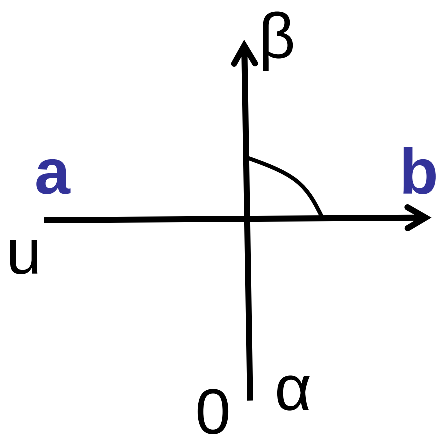
Tsuboi'98 V.K., Sorin,Zabrodin'07

- More general and more transparent with SUSY: new QQ relations.

- Alternative to algebraic Bethe ansatz:
  all the way from R-matrix to nested Bethe Ansatz Equations

- Possible random matrix applications: from characters to matrix integrals.

gl(K|M) super R-matrix and Yang-Baxter

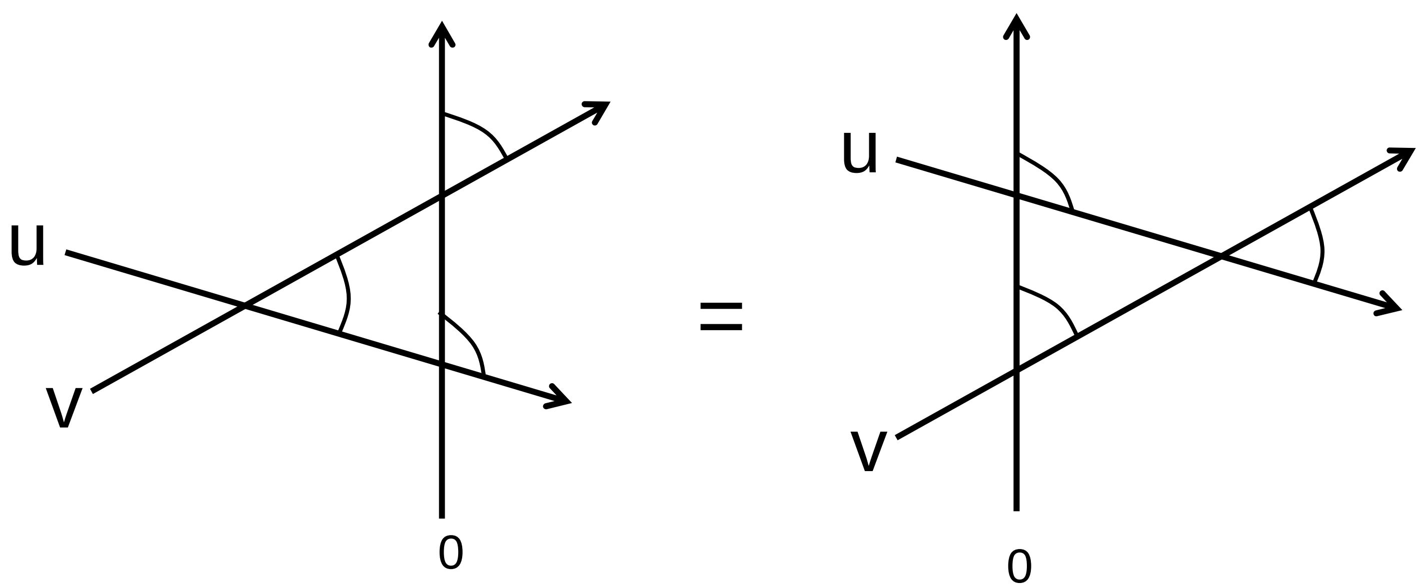


$$R(u) = u + 2\sum_{\alpha\beta}(-)^{p_\alpha} e_{\beta\alpha} \otimes e_{\alpha\beta}$$
$$= u + 2\mathcal{P}$$

(super)permutation

$p_\alpha = 0(1)$; for boson(fermion) index α

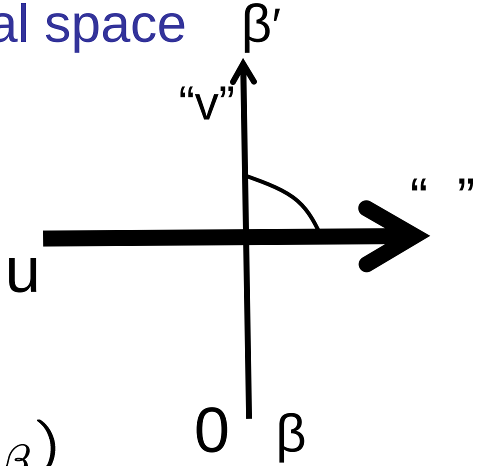
Gl(K|M) generator: $\left(e_{\alpha\beta}\right)^{ab} = \delta_{a\alpha}\delta_{b\beta}$



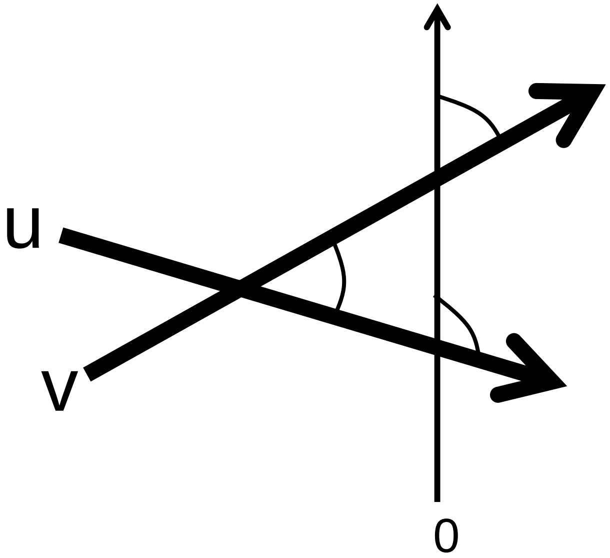
Fused R-matrix in any irrep λ of gl(K|M)

vector irrep "v" in physical space

any " "= { a}
irrep in quantum space



$$R_\lambda(u) = u + 2\sum_{\alpha\beta}(-)^{p_\alpha} e_{\beta\alpha} \otimes \pi_\lambda(e_{\alpha\beta})$$



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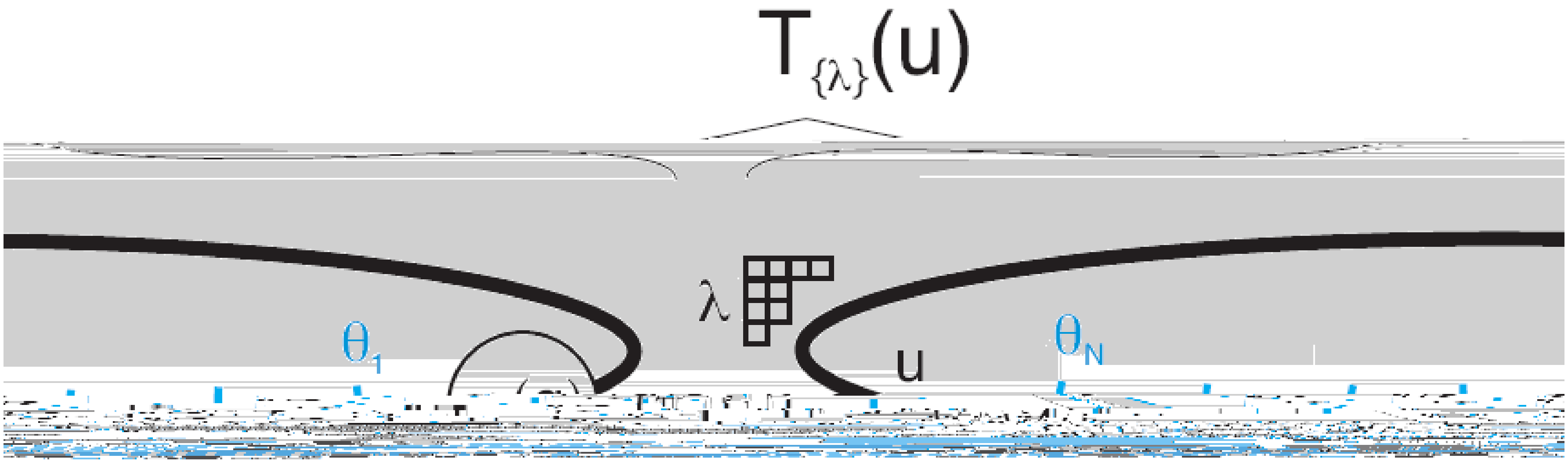
Twisted Transfer Matrix

$$T^{\{\lambda\}}(u) = \mathrm{str}\left(\mathcal{L}^{\{\lambda\}}(u)\pi_\lambda(g)\right)$$

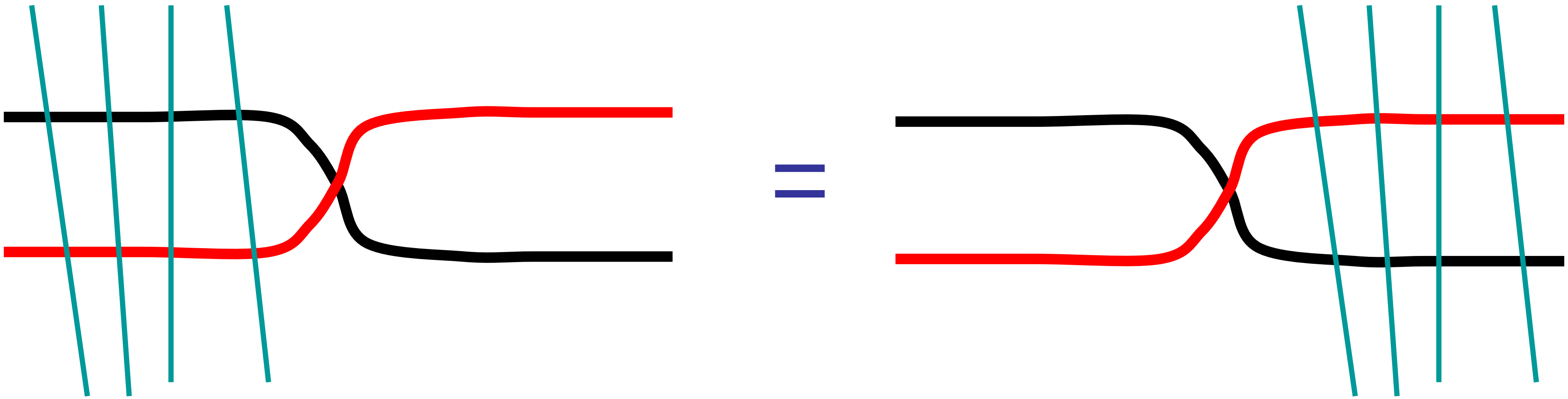
polynomial
of degree N

- Defines all conserved charges of (inhomogeneous) super spin chain:

$$\left[T^{\{\lambda\}}(u), T^{\{\lambda'\}}(u')\right] = 0$$



Commutativity of T-matrices using Yang-Baxter



L(u) L(v) R(u-v)= R(v-u) L(v) L(u)

Bazhanov-Reshetikhin fusion formula

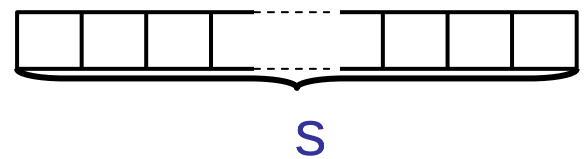
$$T^{\{\lambda\}}(u) = \frac{1}{S_N(u)} \det_{1\le i,j\le a} T_{\lambda_i+i-j}(u+2-2i)$$

Bazhanov,Reshetikhin'90
Cherednik'87

$$S_N(u) = \prod_{n=1}^{N}\prod_{k=1}^{a-1}(u-\theta_a-2k)$$

- Expresses $T^{\{\lambda\}}(u)$ for general irrep $\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_a\}$

  through $T_s(u)$ in symmetric irreps



- Compare to Jacobi-Trudi formula for GL(K|M) characters

$$\chi_{\{\lambda\}}(g) = \det_{1\le i,j\le a} \chi_{\lambda_j+i-j}(g)\,.$$

$\chi_s(g)$ - symmetric (super)Schur polynomials with generating function

$$w(z) \equiv \mathrm{sdet}\,(1-zg)^{-1} = \sum_{s=1}^{\infty} \chi_s(g)\, z^s, \qquad g \in gl(K|M)$$

T-matrix and BR formula in terms of left co-derivative

- Monodromy matrix:

$$\hat{L}^{\{\lambda\}}(u) = (u_1 + 2\hat{D}) \otimes (u_2 + 2\hat{D}) \otimes \ldots \otimes (u_N + 2\hat{D})\pi_\lambda(g), \qquad u_n \equiv u - \theta_n$$

- Transfer-matrix of chain without spins: $T^{\{\lambda\}} = \mathrm{str}\ \pi_\lambda(g) = \chi_{\{\lambda\}}(g)$

- Transfer-matrix of one spin: $T^{\{\lambda\}}(u) = (u + 2\hat{D})\,\chi_{\{\lambda\}}(g)$

- Transfer-matrix of N spins

$$T^{\{\lambda\}}(u) = (u_1 + 2\hat{D}) \otimes (u_2 + 2\hat{D}) \otimes \ldots \otimes (u_N + 2\hat{D})\ \chi_{\{\lambda\}}(g)$$

Proof of BR formula for one spin

Jacobi-Trudi formula for character

$$S(u)\,T^{\{\lambda\}}(u) = \left[\prod_{k=1}^{a-1}(u-2k)\right](u+2\hat{D})\det_{1\le i,j\le a}\chi_{\lambda_i+i-j}(g)$$

should be equal to

$$T^{\{\lambda\}}_{BR}(u) \equiv \det_{1\le i,j\le a}\left[(u+2-2i+2\hat{D})\chi_{\lambda_i+i-j}(g)\right]$$

- First, check for trivial zeroes: every 2x2 minor of two rows

$$T^{\{\lambda\}}_{BR}(2k) = \begin{pmatrix} \cdots & T_{\lambda_k+k-1}(2) & T_{\lambda_{k+1}+k}(0) & \cdots \\ \cdots & T_{\lambda_k+k-2}(2) & T_{\lambda_{k+1}+k-1}(0) & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & T_{\lambda_k+k-a}(2) & T_{\lambda_{k+1}+k-a+1}(0) & \cdots \end{pmatrix}$$

is zero due to curious identity for symmetric characters

$$(1+\hat{D})\chi_{s_1}\,\hat{D}\chi_{s_2} - (1+\hat{D})\chi_{s_1+1}\,\hat{D}\chi_{s_2-1} = 0$$

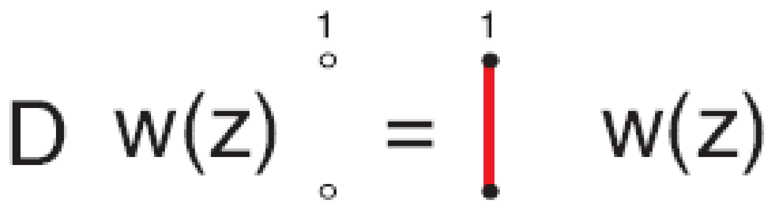
A Hirota-type relation for discrete KdV

Proof of the main identity

- Consider [illegible]

- One derivative

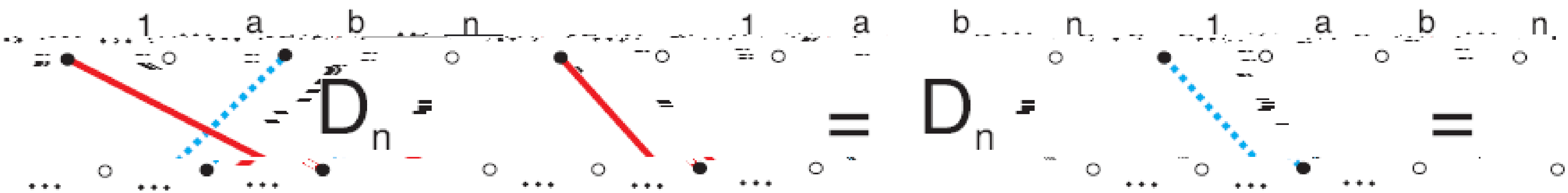
$$\hat{D}\, w(z) = \frac{gz}{1-gz} w(z) \;\; \Leftrightarrow \;\; \left[\hat{D} w(z)\right]^{i_1}_{j_1} = \left[\frac{gz}{1-gz}\right]^{i_1}_{j_1} w(z)$$



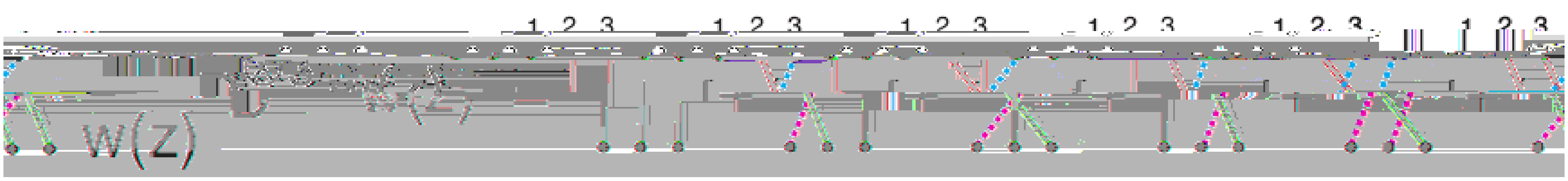
Graphical representation

$$\hat{D} \otimes \frac{gz}{1-gz} = \mathcal{P}\left(\frac{1}{1-gz} \otimes \frac{gz}{1-gz}\right)$$

- One derivative



- Three derivatives



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Proof of the main identity (continuation)

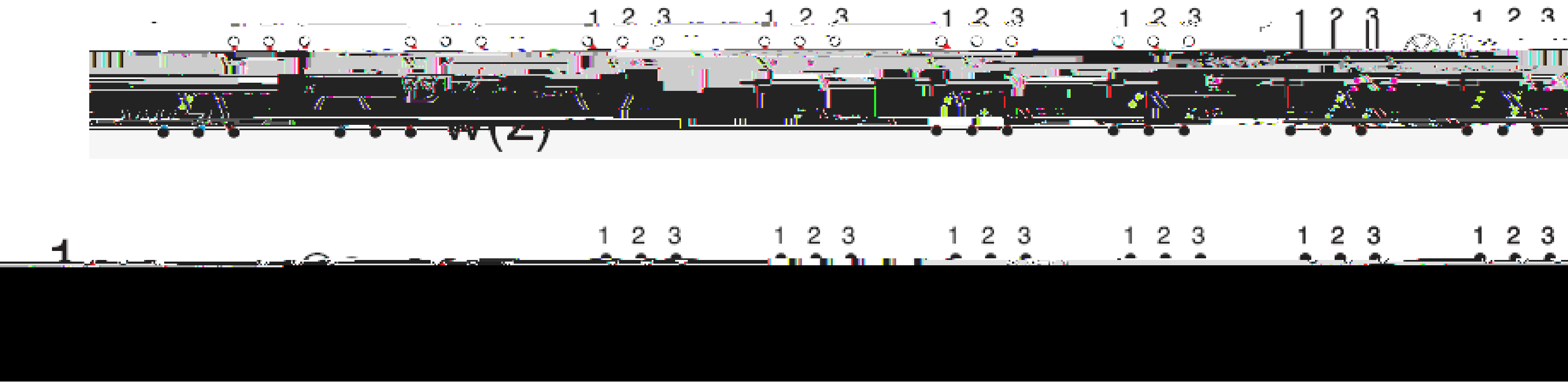
- Consider $$(1+\widehat{D})^{\otimes N} w(z)$$

$$(1+\widehat{D})w(z) = \frac{1}{1-gz}w(z) \quad \Leftrightarrow \quad \left[(1+\widehat{D})w(z)\right]^{i_1}_{j_1} = \left(\frac{1}{1-gz}\right)^{i_1}_{j_1} w(z)$$

- In pictures:

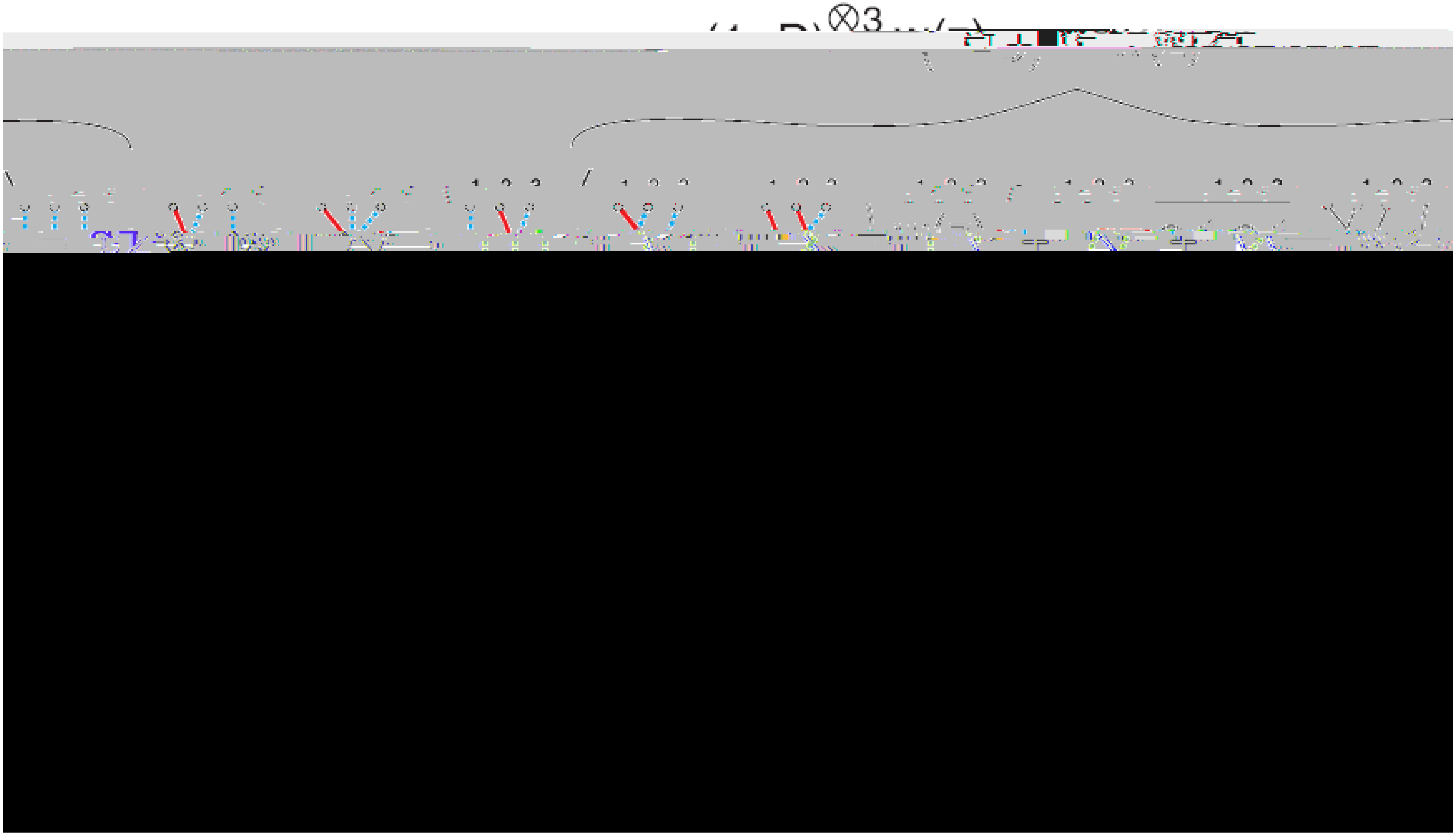
[redacted]

Comparison



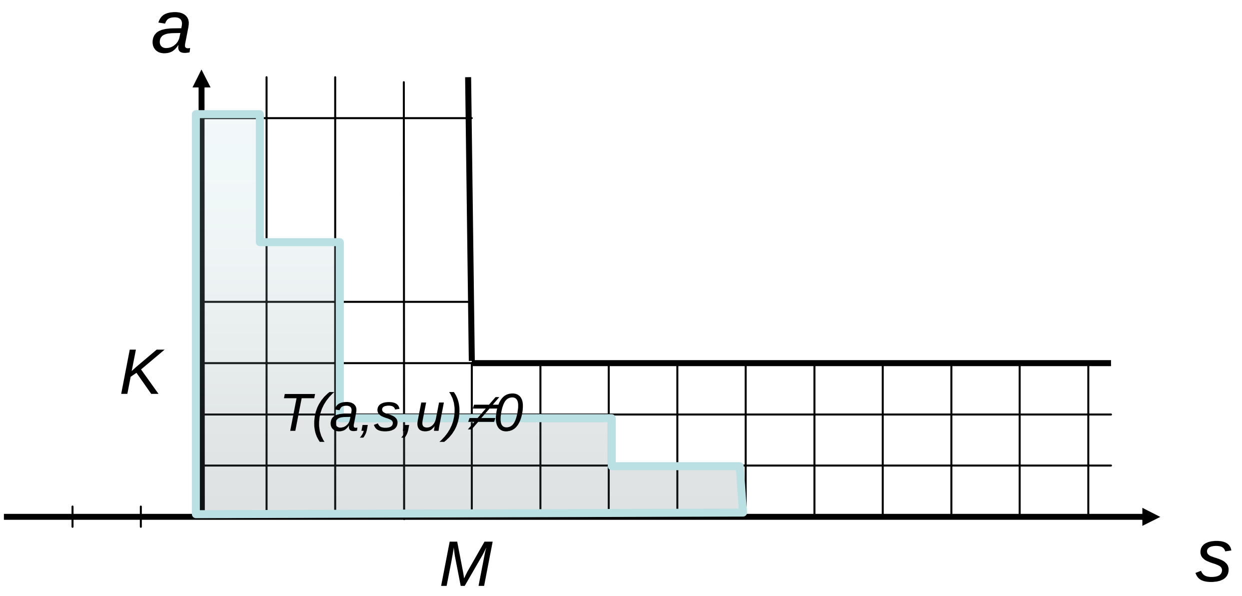
- Notice that the difference is only in color of vertical lines.
- Identical after cyclical shift of upper indices to the right in 2-nd line (up to one line where red should be changed to dotted one)

Proving the identity ….



- This completes our proof of Bazhanov-Reshetikhin formula

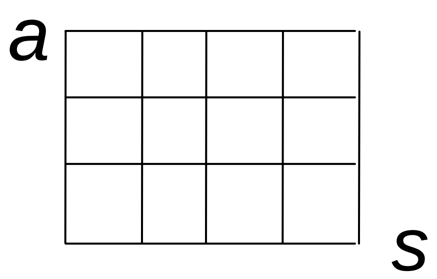
SUSY Boundary Conditions: Fat Hook



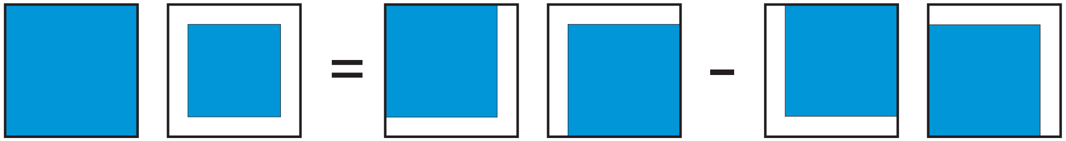
- All super Young tableaux of gl(K|M) live within this fat hook

Hirota eq. from Jacobi relation for rectangular tableaux

T(a,s,u): $\lambda \rightarrow$



- From BR formula, by Jacobi relation for det:



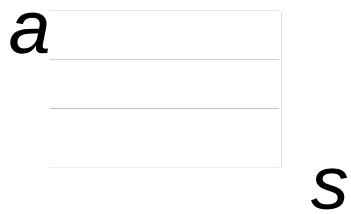
we get Hirota eq.

$$T(a,s,u+1)T(a,s,u-1)-T(a,s+1,u)T(a,s-1,u)=T(a+1,s,u)T(a-1,s,u)$$

- We can solve it by Bäcklund trick, find

From characters to matrix integrals

$$\chi(a,s) = \int \frac{[d\,h]_{\mathrm{SU}(a)}}{(\det h)^{s+1}} \det(1 - h \otimes g)^{-1}$$



or $$Z(g) = \int d^{n^2}M \, \exp \mathrm{Tr}\,[W(gM) + V(M)]$$

or any invariant function of $g \in GL(n)$

- Example: The matrices acting in the space $(\mathbf{V}_n)^{\otimes N}$

$$T^{\{\lambda\}}(u) = (u + u_1 + 2\hat{D}_g) \otimes (u + u_2 + 2\hat{D}_g) \otimes \ldots \otimes (u + u_N + 2\hat{D}_g) \ \langle \mathrm{Tr} \frac{1}{1 - gM} \rangle$$

Commute for any u : $$\left[T(u), T(u')\right] = 0$$

Interesting relations for random matrix correlators!

Solution: Generalized Baxter's T-Q Relations

[V.K.,Sorin,Zabrodin'07]

- Diff. operator generating all T's for symmetric irreps:

- Introduce shift operators:

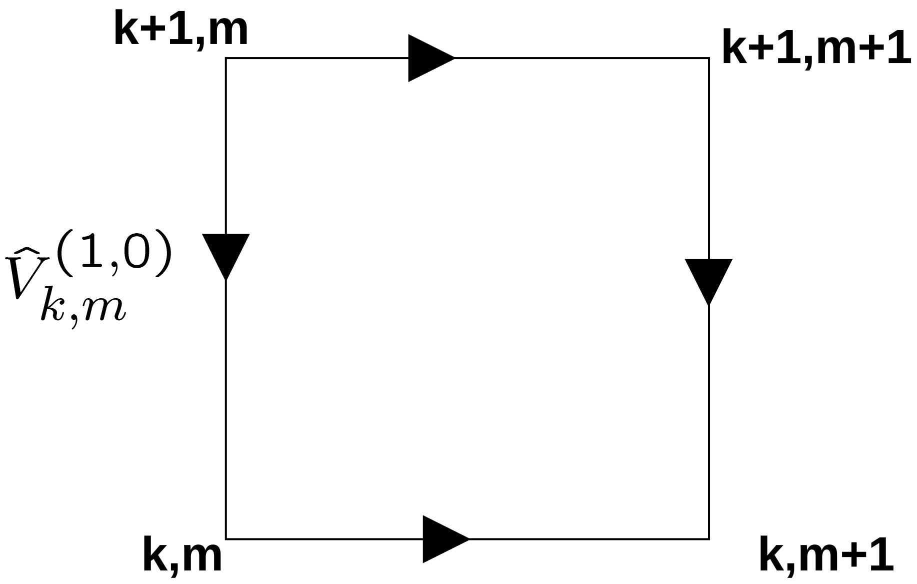
$$\widehat{V}^{(1,0)}_{k,m}(u) = x_k \frac{Q_{k+1,m}(u)\, Q_{k,m}(u+2)}{Q_{k+1,m}(u+2)\, Q_{k,m}(u)} - e^{2\partial_u}$$

$$\widehat{V}^{(0,-1)}_{k,m}(u) = y_m \frac{Q_{k,m}(u)\, Q_{k,m+1}(u+2)}{Q_{k,m}(u+2)\, Q_{k,m+1}(u)} - e^{2\partial_u}$$

Baxter's Q-functions: *Q* *(u-u)*

k=1,…,K
m=1,…M

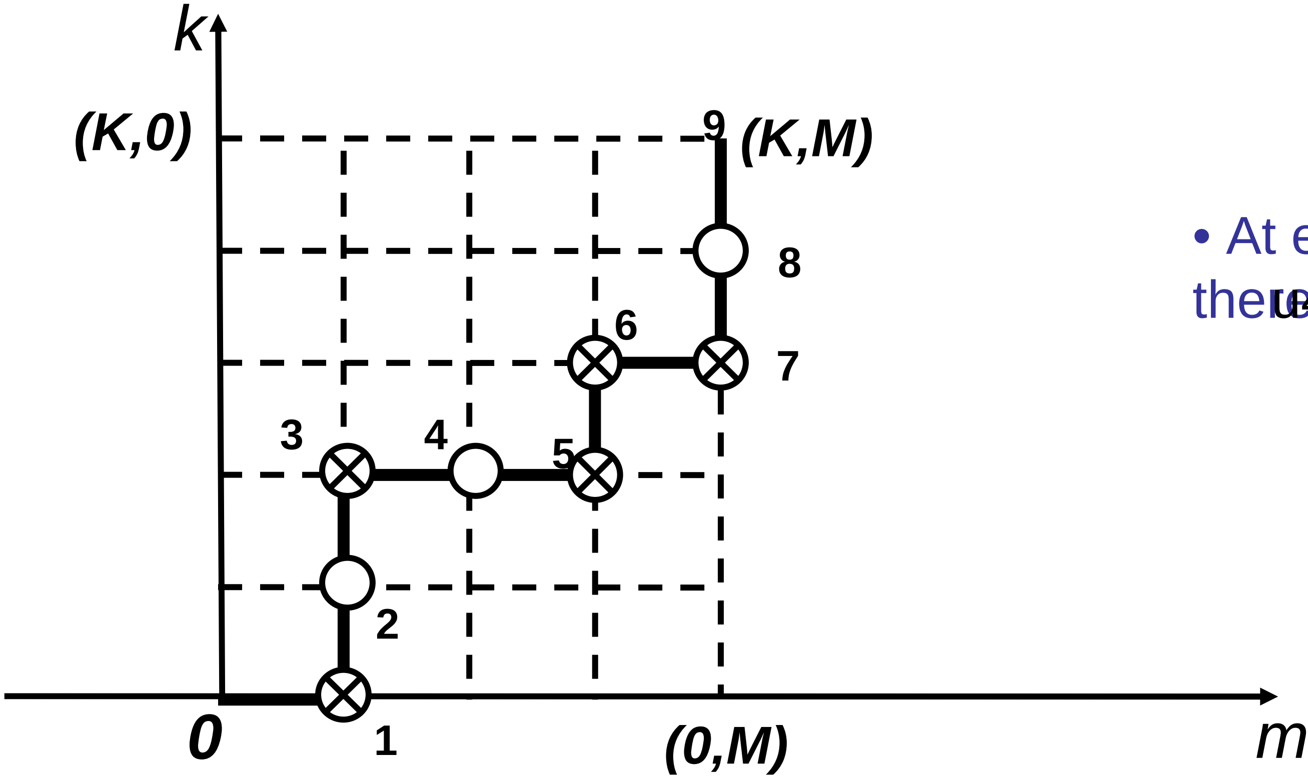
Hirota eq. for Baxter's Q-functions



Zero curvature cond.
for shift operators

$$Q_{k,m}(u)Q_{k+1,m+1}(u+2)-Q_{k+1,m+1}(u)Q_{k,m}(u+2)=Q_{k,m+1}(u)Q_{k+1,m}(u+2)$$

Undressing along a zigzag path (Kac-Dynkin diagram)



• At each (k,m)-vertex there is a u43111111111164sc

Bethe Ansatz Equations along a zigzag path

- BAE's follow from zeroes of various terms in Hirota QQ relation

$$\prod_{b=1}^{K+M} \frac{\check{Q}_b\left(\check{u}_j^{(a)} - K_{ab}\right)}{\check{Q}_b\left(\check{u}_j^{(a)} + K_{ab}\right)} = (-1)^{\frac{1}{2}K_{aa}}\,, \qquad a = 1,\ldots,K+M-1$$

Kulish,Sklianin'80-85

$$Q_{k,m}(u) \underline{= \check{Q}_{k+m}(\check{u})\,, \qquad \check{u}_j^{(n)} = u_j^{(n)} - k + m}$$

and Cartan matrix along the zigzag path

$$K_{ab} = (p_a + p_{a+1})\delta_{a,b} - p_{a+1}\delta_{a+1,b} - p_a\delta_{a,b+1}$$

where $p_n = \begin{cases} 1, \text{ if} \\ -1, \text{ if} \end{cases}$

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Conclusions and Prospects

- We proved Bazhanov-Reshetikhin formula for general fusion.
- We solved the associated Hirota discrete classical dynamics by generalized Baxter T-Q relations, found new Q-Q bilinear relations, reproduced nested TBA eqs. An alternative to the algebraic Bethe ansatz.
- Possible generalizations: noncompact irreps, mixed (covariant+contravariant) irreps, osp(n|2m) algebras. Trigonometric and elliptic(?) case.
- Non-standard R-matrices, like Hubbard or su(2|2) S-matrix in AdS/CFT, should be also described by Hirota eq. with different B.C.
- A potentially powerful tool for studying supersymmetric spin chains and 2d integrable field theories, including classical limits.

$$\frac{T(1,1,u+2)}{\phi(u+3)} = -\frac{Q_{1,0}(u+4)}{Q_{1,0}(u+2)}\frac{\phi(u+2)}{\phi(u+4)} + \frac{Q_{1,0}(u+4)}{Q_{1,0}(u+2)}\frac{Q_{1,1}(u)}{Q_{1,1}(u+2)}\frac{\phi(u+2)}{\phi(u+4)} - \frac{Q_{1,1}(u)}{Q_{1,1}(u+2)}$$