Entanglement entropy in quantum spin chains with nite range interaction

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B

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Fundamentals and Applications
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Work in collaboration with Alexander Its and Man Yue Mo

Outline

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1. The problem

2.

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quantum spin

The problem

Consider a one-dimensional quantum spin chain:

Look at the ground state $|\Psi_g\rangle\langle\Psi_g|$ ($T = 0$: phase transition in the thermodynamic limit.)

Questions we can ask:

- What is the entropy of the entanglement between A and B as $L \to \infty$?
- What is the correlation between two spins at different sites?
- Many others.

If the model is:

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Its, Mezzadri and Mo (2007):

Spin chains and block-Toeplitz determinants

The entropy of the entanglement can be written as

$$S(\rho_A) = \lim_{\epsilon \to 0^+} \lim_{L \to \infty} \frac{1}{4\pi i} \left[\oint_{\Gamma(\epsilon)} e(1+\epsilon; \lambda) \frac{d}{d\lambda} \log D_L(\lambda)(\lambda^2 - 1)^{-L} \, d\lambda \right];$$

Spin chains and block-Toeplitz determinants

Spin chains and block-Toeplitz determinants

$D_L(\quad)$ is the determinant of the block-Toeplitz matrix

$$T_L[\quad] = \begin{pmatrix} B_0 & B_{\ 1} & \cdots & B_{2\ L} & B_{1\ L} \\ B_1 & B_0 & \cdots & B_{3\ L} & B_{2\ L} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{L\ 2} & B_{L\ 3} & \cdots & B_0 & B_{\ 1} \\ B_{L\ 1} & B_{L\ 2} & \cdots & B_1 & B_0 \end{pmatrix}$$

Spin chains and block-Toeplitz determinants

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$$g^2(z) = \prod_{j=1}^{2n} z \quad z_j$$

Spin chains and block-Toeplitz determinants

$$g^2(z) = \quad ^2$$

Spin chains and block-Toeplitz determinants

Remarks:

The branch cuts of $g(z)$ are the segments

$$_i = [\ \ _{2i}\ \ _i ;\ \ _{2i}] ; \quad i = 1;:::;2n$$

$g(z)$ has discontinuities $g_+(z) = \ \ g\ \ (z); \quad z\ 2\ \ \ _i$

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$$L : w^2 = \prod_{i=1}^{4n} (z \qquad _i):$$

The genus of L is $g = 2n \quad 1$.

Statement of results

De ne $: \mathbb{C}^g \to \mathbb{C}$ associated to L by

$$(\vec{s}) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{i \, \vec{n} \quad \vec{n} \;\; 2 \, i \, \vec{s} \quad \vec{n}}.$$

is a $g \quad g$ symmetric matrix (period matrix) that depends on the Hamiltonian through the branch cuts of L.

Statement of results

What happens at a phase transition?

Block-Toeplitz determinants and the RH problem

How do we compute such formulae?

The symbol $(z) =$ i ... $g^{-1}(z)$ ... $g(z)$ ... admi... Wiener-Hopf

$(z) =$

Block-Toeplitz determinants and the RH problem

How do we compute such formulae?

The symbol $(z) = \begin{pmatrix} i & g(z) \\ g^{\ 1}(z) & i \end{pmatrix}$ admits the Wiener-Hopf factorization:

$$(z) = U_{+}(z)U\ \ (z) = V\ \ (z)V_{+}(z);$$

where $U\ \ (z)$ and $V\ \ (z)$ are analytic inside/outside the unit circle and

$$U\ \ (\ 1\ ) = V\ \ (\ 1\ ) = I:$$

Block-Toeplitz determinants and the RH problem

Theorem (Widom, 1974)

$$\frac{d}{d} \log D_L(\;) = \frac{2}{1 \quad {}^2} L + \frac{1}{2} \int_{S^1} \mathrm{tr} \left[\quad U_+' \right.$$

Block-Toeplitz determinants and the RH problem

It turns out that

$$V\ (z) = {}_3U\ (z)^{1}\ {}_3$$
$$V_+(z) = {}_3U_+(z)^{1}\ {}_3(\ ^2\quad 1); \qquad \neq \quad 1$$

Block-Toeplitz determinants and the RH problem

Block-Toeplitz determinants and the RH problem

Let us define

$$
\begin{aligned}
S(z) &= U_-(z)Q(z)^{-1}, && |z| \geq 1,\\
S(z) &= U_+(z)^{-1}Q(z), && |z| \leq 1.
\end{aligned}
$$

In order to compute the entropy of entanglement we need:

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1. to solve the previous RH problem for $S(z)$ in terms of) $\theta 564$

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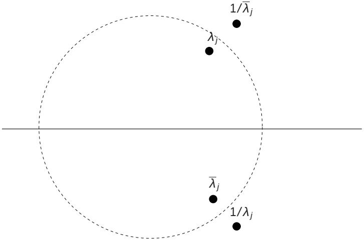
Pairs of roots of $g(z)$ approach the unit circle.



The period matrix in

$$(\vec{s}) := \sum_{\vec{n} \in Z^g} e^{i \, \vec{n} \quad \vec{n} \;\; 2i \;\; \vec{s} \;\; \vec{n}} .$$

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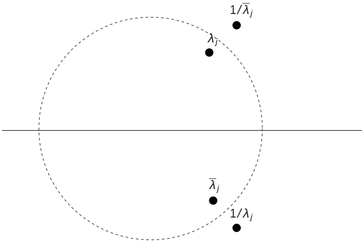

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$(\vec{s})$ becomes singular.

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Such determinants are computed by solving a RH problem.

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- We computed the entropy of the entanglement of the ground state of integrable quantum spin chains with nite range and translation invariant interaction.
- At the core of the computation is the evaluation of block-Toeplitz determinants.
- Such determinants are computed by solving a RH problem.
- At phase transition we observe logarithmic divergences that generalize previous results.

AR Its, F Mezzadri and MY Mo. Entanglement entropy in quantum spin chains with nite range interaction. arXiv:0708.0161v1.