Our activity in this direction was triggered by the paper by Dean and Majumdar, who came up with an approximate theory of large spectral fluctuations in the three Gaussian Random Matrix models – GOE, GUE and GSE.

Non-perturbative Theory of Large Spectral Deviations in GUE

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Large Deviations of Extreme Eigenvalues of Random Matrices

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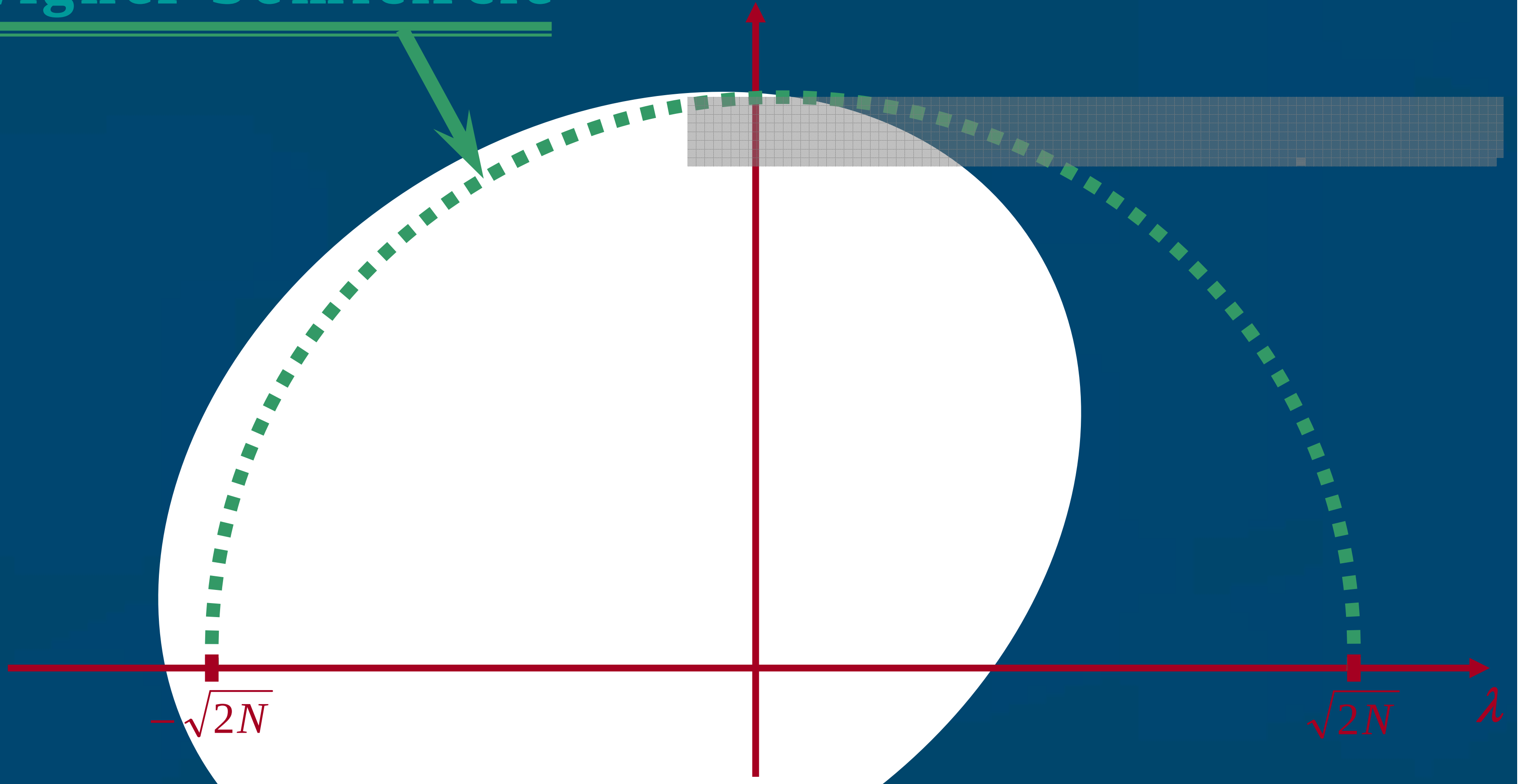
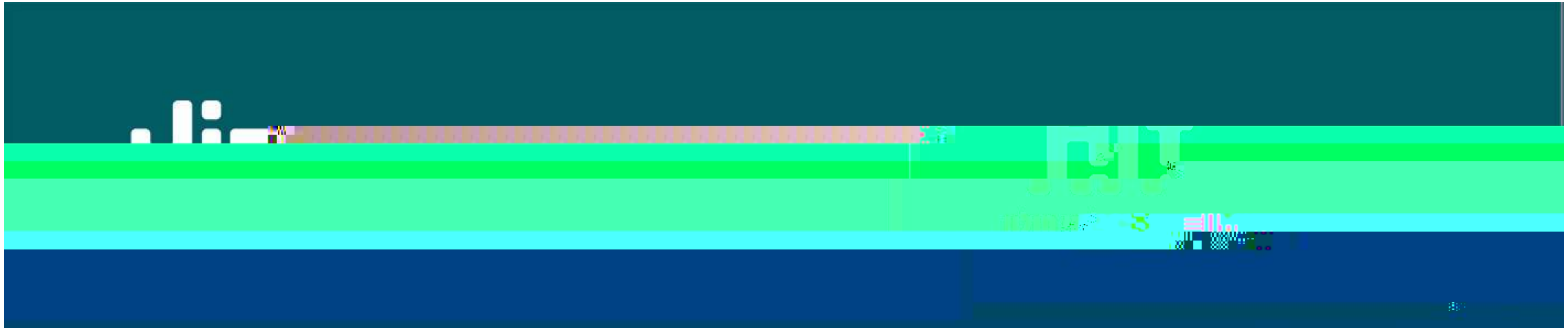
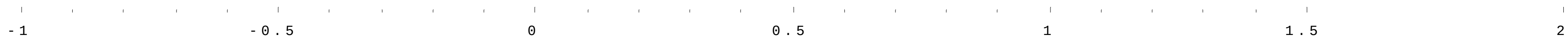
Wigner Semicircle






-1
-0.5
0
0.5
1
1.5
2

The Major Statement of the Work

We managed to find an

D.S. Dean & S.N. Majumdar result $P_N(z) \approx \exp\left(-N^2 S_0(z) + O(N^1)\right)$

GUE_{NxN} $N >> 1$

$$P_N(z) \approx \exp\left(-N^2 S_0(z)\right) R_N(z) \approx \frac{3^{1/8} \tau_0\, S_0(z)}{2^{5/24} N^{1/12}}$$

$$S_0(z) = \frac{(\sqrt{z^2+3}-z)^{1/6}}{(\sqrt{z^2+3}-2z)^{1/8}(z^2+3)^{1/48}}$$

Probability Distribution Function via Coulomb Fluid Approach

An absence of the linear in N term can be appreciated even within the Coulomb fluid approach. Indeed, for general beta, the desired distribution function PN which is defined by a matrix integral can approximately be rewritten as a functional integral. This representation holds in the large N-limit and each term in the action has a clear origin.

$$d\mu_j P_N(z) \sim \int_{-\infty}^{z} |\Delta_N(\boldsymbol{\mu})|^{\beta} \prod_{j=1}^{N} e^{-\beta N \mu_j^2}$$

$$N >> 1$$

$$P_N(z) \sim \int \mathcal{D}[\rho] \exp\left[-\beta N^2 \int_{-\infty}^{z} \frac{\mu^2}{2} \rho(\mu) d\mu \right.$$

$$\frac{\beta N^2}{2} \int \int_{-\infty}^{z} d\mu d\mu' \rho(\mu)\rho(\mu') \ln|\mu - \mu'|$$

$$\left. \int_{-\infty}^{z} d\mu \rho(\mu) \ln[\rho(\mu)] \left(1 - \frac{\beta}{2}\right) N \right]$$

$$\sum_{j=1}^{N} \delta(\mu - \mu_j) \rho(\mu) = \frac{1}{N}$$

Coulomb Fluid Approach

F. Dyson (1962)

1. $e^{-\beta N \sum_j \mu_j^2}$ — $N >> 1$ → $e^{-\beta N^2 \int^z \mu^2 \rho(\mu) d\mu}$

2. $\ldots \ln|\mu_i - \mu_j|$

Short-distance regularization

local level spacing $l(\mu_j) \ldots 1/(N\rho(\mu_j))$

$N >> 1$

$\ldots \approx \frac{\beta}{2} \sum \ln|\mu_i - \mu_j| - \frac{\beta}{2} \sum \ln\left|\frac{1}{N\rho(\mu)}\right|$

$e^{\frac{\beta N^2}{2} \int \int_{-\infty} \ln|\mu - \mu'| \rho(\mu)\rho(\mu') d\mu d\mu' + \frac{\beta N}{2} \int_{-\infty} \rho(\mu) \ln[\rho(\mu)]}$

3. $\int_{-\infty}^{z} \prod_{j=1}^{N} d\mu_j$ — $N >> 1$ →

$$\rho(\mu) = \frac{1}{N} \sum_{j=1}^{N} \delta\left(\mu - \frac{\lambda_j}{\sqrt{2N}}\right)$$

$$\int_{-\infty}^{z} \rho(\mu) d\mu = 1$$

Absence of the Linear in N Term. Coulomb Fluid Approach

At β=2, the linear in N term vanishes. This lead us to conclude that a linear in N term cannot be present in the Dean & Majumdar solution for β=2.

$$P_N(z) \sim \int_{-\infty}^{z} |\Delta_N(\boldsymbol{\mu})|^{\beta} \prod_{j=1}^{N} e^{-\beta N \mu_j^2} d\mu_j$$

$$N >> 1$$

$$P_N(z) \sim \int \mathcal{D}[\rho] \exp\left[-\beta N^2 \int_{-\infty}^{z} \mu^2 \rho(\mu) d\mu \right.$$

$$\left. + \frac{\beta N^2}{2} \int\int_{-\infty}^{z} d\mu\, d\mu' \rho(\mu)\rho(\mu') \ln|\mu - \mu'| \ldots \right]$$

$$\sum_{j=1}^{N} \delta(\mu - \mu_j) \rho(\mu) = \frac{1}{N}$$

Non-perturbative Theory of Large Spectral Deviations

Our consideration is based on the exact representation of the probability distribution function in terms of a solution of the fourth Painleve transcendent. This formula can be derived by using a technique of Fredholm determinant (as was done by Tracy and Widom), or with the help of theory of integrable hierarchies much in line with the works by Adler and van Moerbeke.

$$P_N(z) = \mathrm{Prob}\left[\frac{\lambda_{max}}{\sqrt{2N}} < z\right] \simeq \int^{z} \cdots \prod_{j=1}^{N} \cdots$$

$$P_N(z) = \exp\left(-\int_z^{+\infty} dx\, f_N(x)\right)$$

Painlevé IV in Chazy form

1. **Fredholm determinant representation.**
   C.A. Tracy & H. Widom (1992)

2. **Integrable hierarchies. M. Adler & P. van Moerbeke (2001)**

The appropriate boundary conditions were determined by Tracy and Widom who made us the result by Bassom Clarkson Hicks and McLeod.

$$P_N(x) = \exp\left(- \int_x^{+\infty} [illegible] \, f_N(y) \right)$$

[illegible]

Boundary conditions

A.P. Bassom, P.A. Clarkson, A.C. Hicks, J.B. McLeod (1992)

$$f_N(x)\big|_{x\to+\infty} = 0$$

$$[illegible] \left([illegible] \frac{1}{[illegible]} [illegible] \frac{1}{[illegible]} [illegible] \frac{9}{[illegible]} [illegible] \frac{27}{[illegible]} [illegible] \right)$$

$$[illegible] \, x^{-8} \big) \Big) + N^0 \left(-\frac{1}{16x^2} + \frac{10}{32x^4} + O[illegible] \right)$$

1/N Expansion

Existence of the large parameter N makes it tempting to seek the solution fN in the form of the 1/N expansion. We may argue that Gamma is identically zero. The equations for Lambda and Omega together with appropriate boundary conditions follow directly from the initial equation for fN.

$$f_N(x)|_{x \to +\infty} = 0$$

$$\ldots + \frac{\ldots}{32x^7} + \mathcal{O}(x^{-8})\Big) \quad f_N(x)|_{x \to -\infty} = N^2\Big(-4x - \frac{27}{x} + \frac{\ldots}{2x^3} - \frac{\ldots}{16x^7} \ldots\Big) + N^0\Big(\ldots \frac{1}{\ldots} + \frac{10}{16x^5} \ldots \frac{\ldots}{32x} \ldots\Big)$$

Ansatz

$$f_N(x) = N^2\Lambda(x) + N\Gamma(x) + \Omega(x) + \mathcal{O}(N^{-1})$$

$$F_N(z) \simeq e^{\ldots}$$

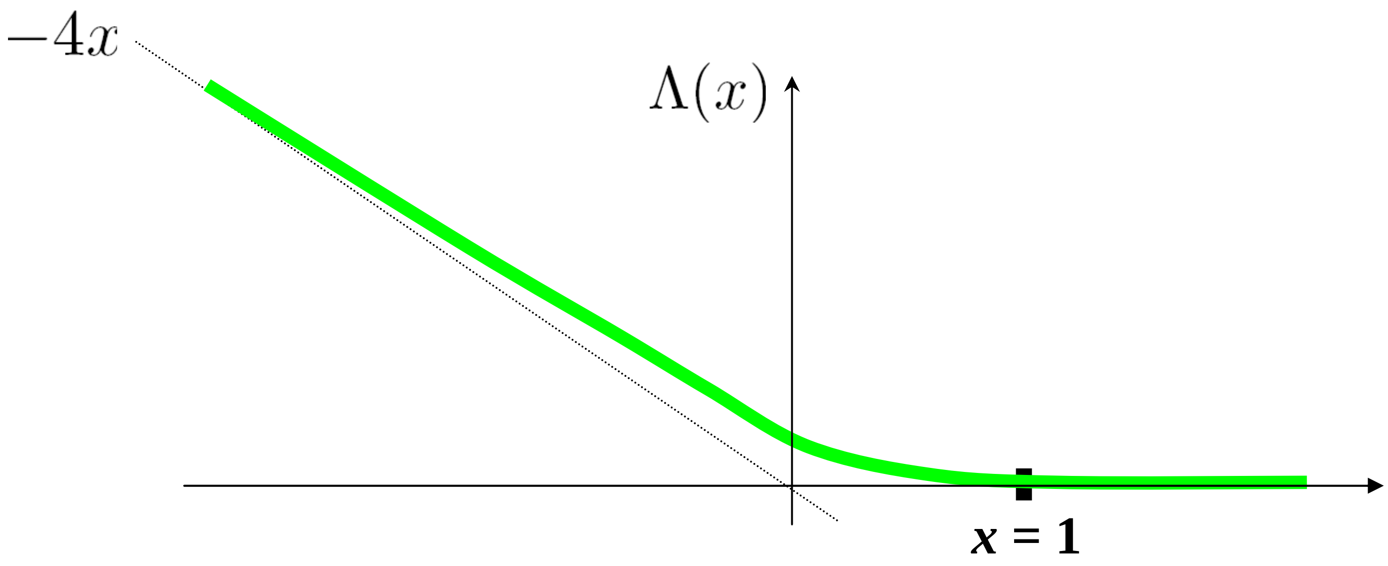
$$6\left(\Lambda'(x)\right)^2 + 16(1 - x^2)\Lambda'(x) + 16x\Lambda(x) = 0$$

$$\Gamma(x) = 0$$

$\Lambda(x)$

The explicit solution for Lambda, that has continuous derivative, is shown on the graph. At minus infinity it approaches the line –4x, as expected. At x large then one, Lambda vanishes.

$$6\left(\Lambda'(x)\right)^2 + 16(1-x^2)\Lambda'(x) + 16x\Lambda(x) = 0$$



Non-perturbativeTheory

$$x = 1 + \frac{s}{2N^{2/3}}$$

$$\rho(s) = \frac{1}{2N^{2/3}} f(1 + s/(2N^{2/3}))$$

Matching with the Tracy-Widom Distribution

$$P_N(z) = \exp\left(-N \int_z^{+\infty} dx\, f_N(x)\right)$$

$$\int_{-cN^{-2/3}}^{+\infty} \cdots \quad \int_z^{-cN^{-2/3}} \cdots \quad \int_z^{-cN} \cdots$$

$$\exp\left(-N^2 \tilde{S}_-(z) \cdots\right)$$

[redacted]

$$\simeq C N \frac{2^{5/24}}{|s|^{1/8}} N^{1/12} e^{-|s|^3/12}$$

$F_2(s)$

$$\simeq \tau_0 \frac{e^{-|s|^3/12}}{|s|^{1/8}}$$

The Major Statement of the Work

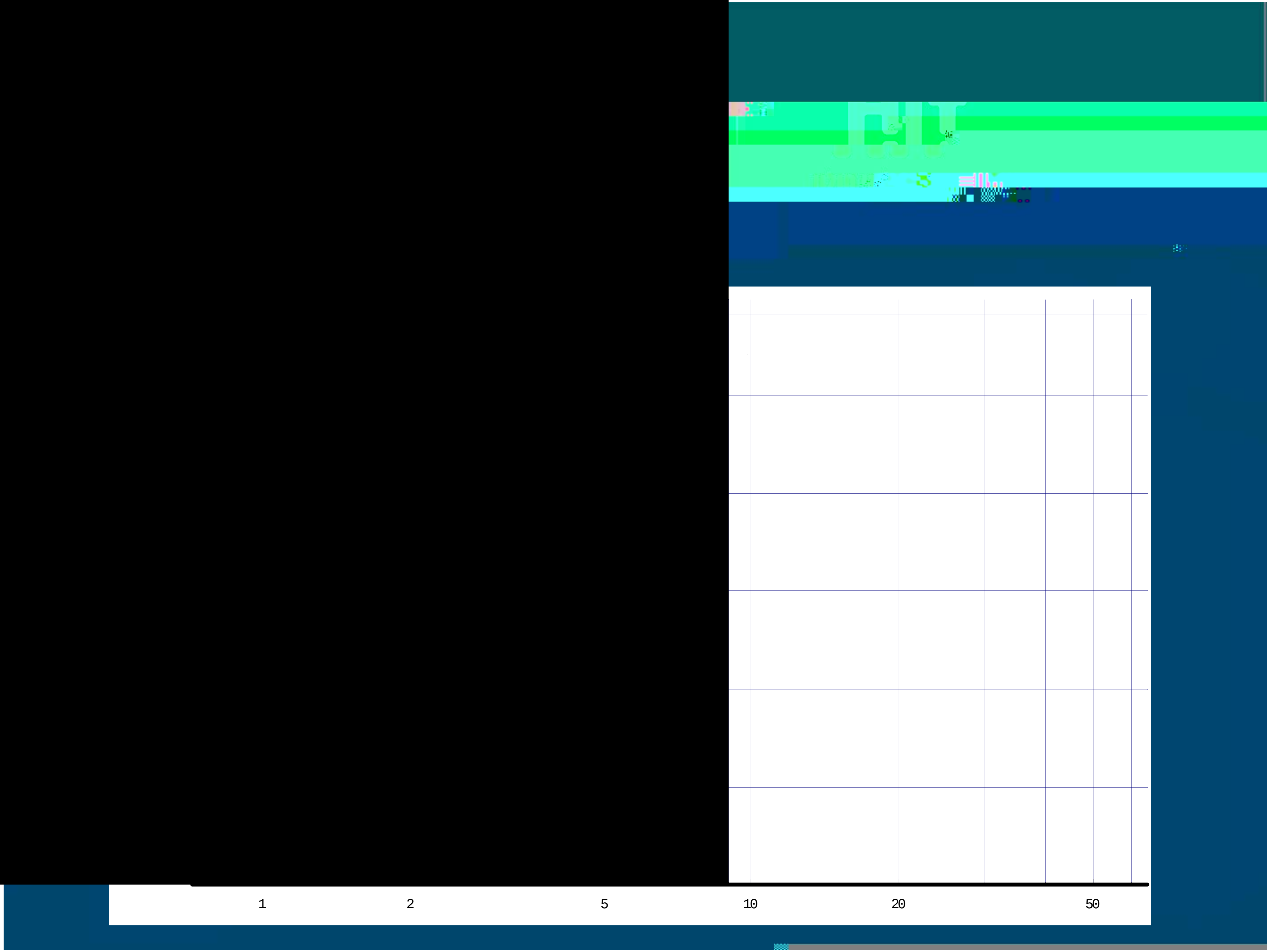
GUE$_{NxN}$ $N >> 1$ $z \in$ **(Large fluctuations domain)**

$$\frac{3^{1/8} \tau_0 \; S_0(z)}{2^{5/24} \; N^{1/12}}$$

***z*=0, *N* = 10**

Numerics :	9.95926x10^{-25}
Dean & Majumdar:	1.39296x10^{-24}
Exact solution:	9.95935x10^{-25}

$$\delta = \left| \frac{Numerics - D\,\&\,M}{Numerics} \right| \sim \mathbf{40}$$


1
2
5
10
20
50

Conclusions

Asymptotically exact solution for the probability of Large Deviations of λ_{max} for $\beta=2$

No linear in N term in the exponent at $\beta=2$

Large fluctuations and Tracy Widom formula are two different scaling limits of the same theory

Similar approach can be developed for $\beta=1$ & 4