Universality of Local Bulk Regime for Hermitian Matrix Models

L. Pastur Ukraine

1 Introduction

1.1 Asymptotic ”Philosophy” of RMT

Let M_n be a

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(iii) $N(\;) = \int \;(\;)d\;.$

Property (ii) fixes the global scale of the spectral axis, yielding

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i.e., $(nN(\;))^{\;1}$ is the typical eigenvalue spacing for large n in .

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Assume $\;\;\;\subset \operatorname{supp} N$, then

  does not depend on n (in the global scale), i.e., $nN(\;\;) \simeq n$: global (macroscopic) regime;

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Probability theory analogs: LLN, CLT, ”collective theorems”, *Yu. Linnik*

”emerging universality” QFT

1.2 Linear Eigenvalue Statistics.

Take $\varphi : \mathbb{R} \to \mathbb{R}$ and write the *linear eigenvalue statistic*

$$N_n[\varphi] := \sum_{l=1}^{n} \varphi(\lambda_l^{(n)}) = \operatorname{Tr} \varphi(M_n) = \int_{\mathbb{R}} \varphi(\lambda) N_n(d\lambda).$$

φ is known as the *test function*. In particular

$$\begin{aligned} N_n(\Delta) &: = \sharp\{\lambda_l^{(n)} \in \Delta, \; l = 1, \dots, n\} \\ &= \sum_{l=1}^{n} \chi_\Delta(\lambda_l^{(n)}) = N_n[\chi_\Delta] \end{aligned}$$

is the Eigenvalue Counting Measure of eigenvalues and $\mathcal{N}_n = n^{-1} N_n$.

Define

$$\text{bulk}\, N = f \quad 2\, \text{supp}\, N : 9 \quad > 0; \lim_{n\,!\,1} \sup_{j} \quad \sup_{j} \quad j\ (\ ) \quad {}_n(\ )j = 0g:$$

We have for $N_n[\,'\,]$:

- $'$ is n-independent: global regime;
- $'_n = '((\qquad {}_0)L_n); \ L_n \ ! \ 1; \ nL_n \ ! \ 0$: intermediate bulk regime;
- $'_n = '((\qquad {}_0)n \ {}_n(\ {}_0))$: local bulk regime

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(ii) $\mathrm{Var}\{N_n\}$, global and intermediatee regime, fluctuations, ”CLT”

(iii) $\mathrm{P}\{N_n(\ _n) = k\};\ k \in \mathbb{N};\ \ E_n(\ _n) = \mathrm{P}\{N_n(\ _n) = 0\}$,

gap probability, local regime, spacings, universality

in particular $\ _n = (\ _0;\ \ _0 + s/n\ \ _n(\ _0));\ \ \ _0 \in \text{bulk } N$ for the local bulk regime

1.4 Hermitian Matrix Models

$n \times n$ hermitian random matrices with the law

$$\mathbb{P}_n(dM) = Z_n^{-1} \exp\{-n \operatorname{Tr} V(M)\} dM;$$

$$dM = \prod_{j=1}^{n} dM_{jj} \prod_{1 \le j < k \le n} d\Re M_{jk} \, d\Im M_{jk};$$

$V : \mathbb{R} \to \mathbb{R}_+$ is a continuous function (potential), and

$$\exists \, \varepsilon > 0; \; L < \infty \quad V(\lambda) \ge (2 + \varepsilon) \log(1 + |\lambda|) > 0; \; |\lambda| \ge L$$

$V = \lambda^2/2$ corresponds to the Gaussian Unitary Ensemble (GUE).

1.5 Results (a collection)

(i) For any probability measure m on $\mathrm{R};\ m(\mathrm{R}) = 1$ define (*Gauss*)

$$E[m] = \int V(\;)m(d\;) \qquad \int\int \log j \qquad j m(d\;)m(d\;);$$

and let N

In particular

$$\mathrm{v.p.}\int_{\mathrm{supp}\ N}\frac{(\quad)d}{} = V'(\quad)/2; \quad 2\ \mathrm{supp}\ N.$$

i.e., an analog of the LLN : *Wigner, 52; Brezin et al, 79; A. Boutet de Monvel, P., Shcherbina, 95; Deift et al 98; Johansson, 98; P., Shcherbina, 07*.

(ii) $\mathrm{Var}\, f N_n[\ '\ ]g$ does not grow with n if $'\ \ 2$

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V' *is Lip 1;*

there exists a closed interval $[a, b] \quad = \operatorname{supp} N$ *such that*

$$\sup_{\in [a,b]} |V'''(\ )| \quad C_1 < \infty, \quad 0 < \inf_{\in [a,b]} \ (\ ).$$

Then we have for any $d > 0$:

(i)

$$\sup_{\in [a+d, b \quad d]} | \ _n(\ ) \quad (\ )| \quad Cn^{\ 2/9};$$

i.e., $[a + d, b \quad d]$ *belongs to bulk* N *;*

(ii) if $p_l^{(n)}; \; l = 1; 2; \ldots$ *are the marginals of the joint probability density of eigenvalues, then for any* $\;_0 \in [a + d; b \quad d]$

$$\lim_{n \to \infty} [\;_n(\;_0)]^{\;l} p_l^{(n)} \quad \;_0 + \frac{x_1}{n \;_n(\;_0)}; \ldots; \quad \;_0 + \frac{x_l}{n \;_n(\;_0}$$

(iii) *if* $E_n(\Delta_n) = \mathrm{P}\{\lambda_l^{(n)} \notin \Delta_n;\ l = 1, \ldots, n\}$ *is the gap probability of the ensemble and* $\Delta_n = (\lambda_0, \lambda_0 + s/n\rho(\lambda_0))$, $\lambda_0 \in [a + d, b - d]$, *then*

$$\lim_{n \to \infty} \mathrm{P}(\Delta_n) = \det(1 - S(s));$$

where

$$(S(s)f)(x) = \int_0^s \frac{\sin \pi(x - y)}{\pi(x - y)} f(y)dy; \quad x \in [0, s].$$

Dyson, 61, 73; P., Shcherbina, 97, 07; Deift et al 99

More if $\rho(\lambda_0) = 0; \infty$ (singular points (e.g. edge) universality).

2 Proof (outline)

2.1 Orthogonal Polynomials Techniques

Weyl integration formula for the joint eigenvalue density

$$p_n(\ _1,\ldots,\ _n) \quad = \quad Q_n^{\ 1} e^{\ n \sum_{k=1}^{n} V(}$$

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$$p_n$$

$\;_l^{(n)} = e^{\;nV/2}P_l^{(n)}$ and $K_n(\;;\;) = \sum_{l=0}^{n\;1} \;_l^{(n)}(\;)\;_l^{(n)}(\;)$,

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$$\int K_n(\;;\;)K_n(\;;\;)d\; = K_n(\;;\;):$$

Then the marginals $p_l^{(n)}$ of p_n are given by the determinant formulas

$$p_l^{(n)}(\;_1;:::;\;_l) := \int p_n(\;_1;:::;\;_n)d\;_{l+1}:::d\;_n$$

$$= (n:::(n\quad l+1))^{\;1}\det\{K_n(\;_j;\;_k)\}_{j;k=1}^{l}:$$

Determinant formulas imply:

(a) $\mathbb{E}\{n^{\;1}N_n[\,']\} = \int '(\;)\;_n(\;)d\;;\quad _n(\;) = K_n(\;;\;);$

(b) $\mathrm{Var}\{N_n[\varphi]\} = \frac{1}{2}\int\int (\varphi(\theta_1) - \varphi(\theta_2))^2 K_n^2(\theta_1, \theta_2)\, d\theta_1 d\theta_2;$

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(c) $\mathrm{P}\{N_n(\quad) = 0\} = \det(1 \quad K_n(\quad))$, where

$$(K_n(\quad)f)(\ ) = \int K_n(\ ;\ )f(\ )$$

These asymptotics were found by *Deift et al, 99* and lead to the solution of (iii), i.e., the universality of the local bulk regime of the hermitian matrix models, and (ii), i.e., the limiting laws of fluctuations of linear eigenvalue statistics, the CLT in particular, *P., 06*, for real analytic potentials.

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On the other hand, in *P., Shcherbina 97, 07* the universality of the local bulk regime of hermitian matrix models is proved for globally C^2 and locally C^3 potentials (see above theorem), basing on the orthogonal polynomial techniques, in particular on the above integral representation for K_n, but NOT using asymptotics of corresponding orthogonal polynomials.

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In *P., Shcherbina, 97* $\sin(\ x)=(\ x)$ is obtained via its Taylor expansion.

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2.2 Integro-differential Equation for Rescaled Reproducing Kernel

We start from the integral representation à la determinant formulas

$$n^{\ 1} K_n(\ ;\ ) = Q_{n;2}^{\ 1} e^{\ n(V(\ )+V(\ ))=2} \int \prod_{j=2}^{n} d\ _j e^{\ nV(\ _j)} (\qquad _j)(\qquad _j) \prod_{2\ \ j<k\ \ n} (\ _j \qquad _k)^2$$

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$$\frac{\partial}{\partial x}K_n(x;y) = \frac{1}{2}V'({}_0 + x/n)K_n(x;y) + \int \frac{K_n(x';x')K_n(x;y) - K_n(x;x')K_n(x';y)}{x - x'}\,dx'.$$

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We prove next that under the conditions of theorem we have uniformly in $|x|, |y| < L,\ \theta_0 \in [a + d, b - d]$:

$$\left|\frac{\partial}{\partial x}K_n(x,y) + \frac{\partial}{\partial y}K_n(x,y)\right| \le C\left(n^{-1/8} + |x - y|n^{-2}\right);$$

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$$|K_n(x,y) - K_n(0,y-x)| \le C|x|\left(n^{-1/8} + |x-y|n^{-2}\right),$$

$$\left|\frac{\partial}{\partial x} K_n(x,y)\right| \le C, \quad \int_{|x|\le L} dx \left(\frac{\partial}{\partial x} K_n(x,y)\right)^2 \le C.$$

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$$\left|\frac{\partial}{\partial x} K_n(x,y)\right| \le C, \quad \int_{|x| \le L} dx \left(\frac{\partial}{\partial x} K_n(x,y)\right)^2 \le C.$$

Now, if

$$K_n(x) = K_n(x;0)\mathbf{1}_{|x|\le L} + K_n(L;0)(1+L-x)\mathbf{1}_{L<x\le L+1} + K_n(-L;0)(1+L+x)\mathbf{1}_{-L-1\le x<-L};$$

then for $|y| \le L$

$$\frac{\partial}{\partial y} K_n(y) = \int_{|x'|\ge 2L/3} \frac{K_n(x')K_n(y-x')}{x'}\,dx' + O(L^{-1});$$

and $\displaystyle\int |K_n(x)|^2 dx \lesssim 1\,;\quad \int \left(\frac{d}{dx}K_n(x)\right)^2 dx \lesssim 1\,.$

2.3 Asymptotic Solution of Equation

Consider the Fourier transform

$$\widehat{K}_n(p) = \int K_n(x) e^{ipx} dx; \quad K_n(x) = (2\ \ )^{\ 1} \int \widehat{K}_n(p) e^{\ \ ipy} dp.$$

Then we have from $n\ \ ^{1} K_n(\ \ ;\ \ ) =$

Since K_n is "asymptotically even"

$$\int |\hat{K}_n(p) - \hat{K}_n(-p)|^2 dp$$
$$= 2\pi \int |K_n(x) - K_n(-x)|^2 dx \le C n^{-1/8} \log^3 n.$$

we obtain the Fourier form of the above integro-differential equation:

$$\int \hat{K}_n(p) \int_0^p \hat{K}_n(p')dp' \, p \, e^{-ipy} dp = O(L^{-1}); \quad |y| \ge L/3.$$

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we obtain the Fourier form of the above integro-differential equation:

$$\int \hat{K}_n(p) - \int_0^p \hat{K}_n(p')dp' \; p \; e^{-ipy} dp = O(L^{-1}); \; |y| \le L/3.$$

Besides, since K_n is positive definite $\hat{K}_n$ is ”asymptotically non-negative”:

$$\int \hat{K}_n(p)|\hat{f}(p)|^2 dp \ge -C\|f\|^2_{L^2(\mathbb{R})}(n^{-1/8} \log^4 n + O(L^{-1})).$$

Now consider the functions

$$F_n(p) = \int_0^p \hat{\kappa}_n(p')\,dp'.$$

Since $p\hat{\kappa}_n \in L^2(\mathbb{R})$, the sequence $\{F_n\}$ consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on $\mathbb{R}$.

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(b) $F(p) = -F(-p)$;

(c) $F(p) \leq F(p')$, if $p \leq p'$;

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$$F_n(p) = \int_0^p \tilde{R}_n(p')dp'.$$

Since $p\tilde{R}_n$

(e) F satisfies the following equation, valid for any smooth function g of compact support:

$$\int (F(p) \qquad p)g(p)dF(p) = 0.$$

The last property implies that $F(p) = p$ or $F(p) = \text{const}$, hence it follows from (a) – (c) that

$$F(p) = p\,1_{|p| \quad p_0} + \qquad \text{sign}(p)\,1_{|p| \quad p_0};$$

where $p_0 = \qquad (\ \ _0)$ by (d).

We conclude that the equation is uniquely soluble, thus the sequence $\{F_n\}$ converges uniformly on any compact to the above F. This imply the weak convergence of the sequence fK_ng to the function

$$K\ (x) = \frac{\sin(\quad (\ _0)x)}{(\ _0)x}:$$

convergence of the sequence $f K_n g$ to the function

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But weak convergence implies

$$\lim_{n \ ! \ 1} K_n(x;y) = K \ (x \quad y):$$

uniformly in $(x;y)$, varying on a compact set of R^2, because

$$\frac{d}{dx}K_n \ 2 \ L^2(\mathrm{R}):$$