

# Exact Minimum Eigenvalue Distribution of a Random Entangled Bipartite System

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## *Collaborators:*

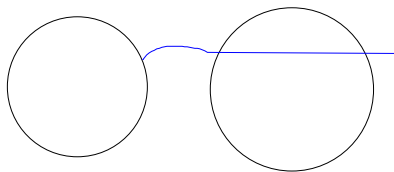
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Ref: [J. Stat. Phys. 131, 33 \(2008\)](#)

## Plan:

- A brief review of the physical system
  - = Randomly coupled entangled bipartite quantum system
- Reduced density matrix and its eigenvalue statistics
- Minimum eigenvalue  $\min$  exact PDF  
proving, on the side, a recent conjecture by [Znidaric](#) (2007).
- Summary and Conclusions

# Coupled Bipartite System



Composite System:  $A \otimes B$

Any state:

$$|\psi\rangle = \sum_i x_i |i_A\rangle \otimes |i_B\rangle$$

$X = [x_i]$  ( $N \times M$ ) rectangular Coupling matrix

# Coupled Bipartite System

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$$| \psi \rangle = \sum_i x_i |i_A \rangle | B \rangle$$

- If  $x_i = a_i b$  then

$$| \psi \rangle = \sum_i a_i |i_A \rangle b | B \rangle = | A \rangle | B \rangle$$

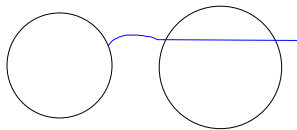
– Fully Un-entangled (factorised)

Otherwise – Entangled (non-factorisable)

- Density matrix of the composite system

$$\rho_{AB} = \sum_{i,j} x_i x_j^* |i_A \rangle \langle j_A \rangle | B \rangle \langle B \rangle$$

# Reduced Density Matrix of subsystem A:



- Reduced Density Matrix:  $\hat{\rho}_A = \text{Tr}_B[\hat{\rho}]$  with  $\text{Tr}[\hat{\rho}_A]$



# Summary:

- Schmidt decomposition:

$$|\psi\rangle = \sum_{i=1}^N \sqrt{\lambda_i} |i\rangle^A |i\rangle^B$$

- Reduced density matrix:  $\hat{\rho}_A = \text{Tr}_B [|\psi\rangle\langle\psi|] = \sum_{i=1}^N \lambda_i |i\rangle^A \langle i|^A$

- $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$  eigenvalues of  $W = XX^\dagger$  with

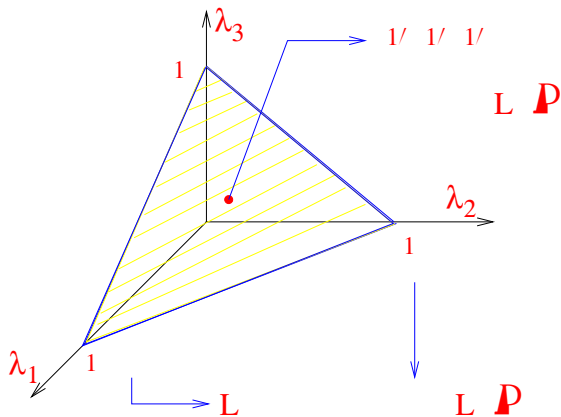
$$0 \leq \lambda_i \leq 1 \quad \text{and} \quad \sum_{i=1}^N \lambda_i = 1$$

- Least** entangled (unentangled): if  $\lambda_1 = 1$  and  $\lambda_2 = \lambda_3 = \dots = \lambda_N = 0$

$$|\psi\rangle = |1\rangle^A |1\rangle^B \quad (\text{fully factorised})$$

- Most** entangled: if  $\lambda_1 = \lambda_2 = \dots = \lambda_N = 1/N$

# A Simple Diagram for $N=3$





# Minimum Eigenvalue $\lambda_{\min}$ :

- Another important object:  $\min =$

# Randomly Coupled Bipartite System

$$| \psi \rangle = \sum_i x_i |i_A\rangle |i_B\rangle = \sum_{i=1}^N \frac{1}{\sqrt{2}} |i\rangle^A |i\rangle^B$$

- $X = [x_i]$  entries are **Random** variables drawn from:

$$\text{Prob}[X] = \exp \left[ -\frac{1}{2} \text{Tr} X^\dagger X \right]; \quad d = 2 \text{ (complex)} \text{ and } d = 1 \text{ (real)}$$

- $\{x_i\}$  **random** variables

- entropy  $S = - \sum_{i=1}^N \frac{1}{2} \ln \left( \frac{1}{2} \right)$  **random** variable

# Znidaric Conjecture and our Results:

- For  $M = N$  and  $k = 2$ , Znidaric (2007) conjectured:

$$\min_{\mathcal{A}} \frac{1}{N^3} \text{ for all } N$$

- For  $M = N$  and

# Minimum Eigenvalue Distribution for the Real Case

- For  $M = N$  and  $\beta = 1$  we prove: for  $0 < x < 1/N$

$$P_N(x) = A_N x^{-N/2} (1 - Nx)$$

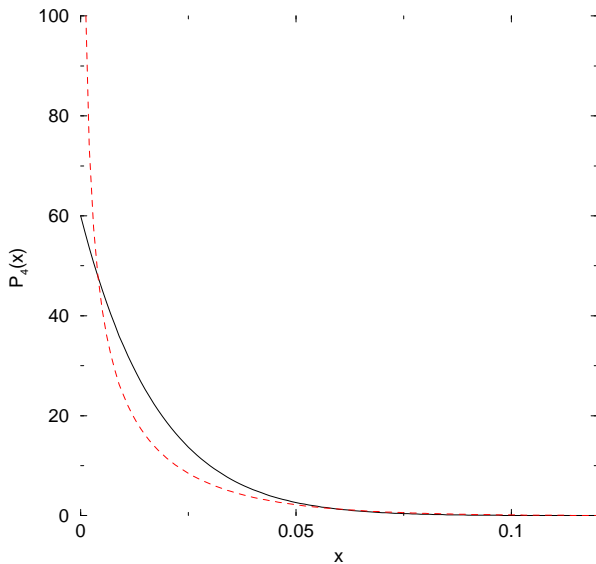
where

$$A_N = \frac{N(N-1)(N-2)}{2^{N-1} (N/2) ((N^2 + N - 2)/2)}$$

- For example for  $N = 2$ :  $P_2(x) = \frac{1-2x}{x(1-x)}$  for  $0 < x < 1/2$
- All moments calculated explicitly: For example the average:

$$\mu_1(N) = \int_0^{1/N} x P_N(x) dx = \frac{1}{N}$$

# Exact Minimum Eigenvalue PDF for $N=4$





# Cumulative distribution of $\lambda_{\min}$ for $M=N$

- For  $M = N$  case

$$Q_N(x) = B_N \prod_{i=1}^N x^{i-1} \prod_{j < k} \frac{d_j}{d_k} \prod_{i=1}^N d_i^{\frac{1}{2}-1}$$

- Integral representation

$$x^{i-1} = \frac{ds}{2i} e^{s(1-P_i)}$$

- $Q_N(x) = B_N \frac{ds}{2i} e^s I_N(x, s)$  where

$$I_N(s, x) = \prod_{i=1}^N x^{i-1} \prod_{j < k} \frac{d_j}{d_k} \prod_{i=1}^N d_i^{\frac{1}{2}-1} e^{-s P_i}$$

# $M = N$ and $\beta = 2$ Case

- For  $\beta = 2$  the integral simplifies:

$$I_N(s, x) = \int_0^x \dots \int_0^x e^{-s \sum_{j < k} (j - k)^2} \prod_{i=1}^N dz_i$$

- Using a shift:  $z_i = s(z_i - x)$  Selberg Integral

$$I_N(s, x) = \frac{e^{-sNx}}{s^{N^2}} \prod_{j=0}^{N-1} (j+2)(j+1)$$

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# Summary and Conclusions:

- **Exact** PDF of the minimum eigenvalue  $\lambda_{\min}$  of the reduced density matrix of a randomly coupled bipartite system of equal sizes for  $\beta = 1$  and  $\beta = 2$

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