Extreme Value Statistics of 1/f Noises generated by 2d Gaussian Free Field:

Statistical Mechanics Approach

Yan V Fyodorov

School of Mathematical Sciences
The University of Nottingham, UK

19 December 2009, Brunel University, West London

References:

Y V F, J-P Bouchaud : J. Phys.A: Math.Theor 41 (2008), 372001 (12pp)

Y V F, P Le Doussal , and A Rosso : J. Stat. Mech. (2009) P10005 (31pp)

Summary of standard extreme-value statistics:

- Let $z_1, \dots, z_N$ be i.i.d. random variables with probability density function $p(z)$. Let $y_N = \max_{i=1,\dots,N}\{z_n\}$ be **maximum** of the set, and $F_N(y) = Prob(y_N < y)$ be the **distribution of the maximum**. Then for $N \gg 1$ the distribution approaches a **scaling form** $F_N(y) \approx F_\infty[(y + a_N)/b_N]$ where a_N, b_N depend on $p(z)$ but the shape of F_∞ is **universal**, and given by

$$F_\infty(y) = \begin{cases} e^{-e^{-y}}, & \forall y \\ e^{-y^{-\alpha}}, & y \geq 0 \\ e^{-[-y]^{\alpha}}, & y \leq 0 \end{cases}$$

Gumbel class: $z < \infty$ and $p(z \to \infty) \sim A e^{-z^{\alpha}}$; $\alpha > 0$

Fréchet class: $z < \infty$ and $p(z \to \infty) \sim A z^{-(\alpha+1)}$

Weibull class: $z < a$ and $p(z \to a) \sim A(a - z)^{(\alpha - 1)}$

The result is rather robust if variables are short-range correlated. In particular, for Gaussian-distributed variables with $\langle z_i \rangle = 0$ the **Gumbel** distribution is known to be valid as long as $C(|i-j|) = \langle z_i z_j \rangle \lesssim const/\ln|i-j|$ for $|i-j| \gg 1$.

Very few explicit results exist for extrema of **strongly correlated variables**, as e.g. for Brownian motion by **Majumdar & Comtet**, or for the largest eigenvalues of random matrices by **Tracy & Widom**.

Gaussian Free Field: definition:

- Given any domain $\mathbf{D}$ consider the **Laplace** operator $-\Delta$ and denote $\mathbf{e}_j(\mathbf{x})$ and $\lambda_j > 0$ for $j = 1, 2, \ldots, \infty$ its eigenfunctions/eigenvalues corresponding to the Dirichlet boundary conditions. Then the functions $\tilde{\mathbf{e}}_j(\mathbf{x}) = \frac{1}{\sqrt{\lambda_j}} \mathbf{e}_j(\mathbf{x})$ form an orthonormal basis of the Hilbert space H w.r.t. the so-called **Dirichlet** scalar product

$$(f, g) = \int_{\mathbf{D}}$$

Gaussian Free Field: examples:

- **GFF** on the interval $\mathbb{D} = [0,1]$. Eigenf./eigenv. for the Laplacian $\Delta = -\frac{d^2}{dx^2}$ (Dirichlet b.c.): $e_n(x) = \sqrt{2}\sin n\pi x,\ \lambda_n = \pi^2 n^2$. The corresponding GFF is given by the random Fourier series $V(x) = \sum_{n=1}^{\infty} \xi_n \frac{\sqrt{2}}{\pi n}\sin n\pi x$, with the covariance given by the Green function $G(x_1, x_2) = \min(x_1, x_2)[1 - \max(x_1, x_2)]$ - **Brownian bridge**.

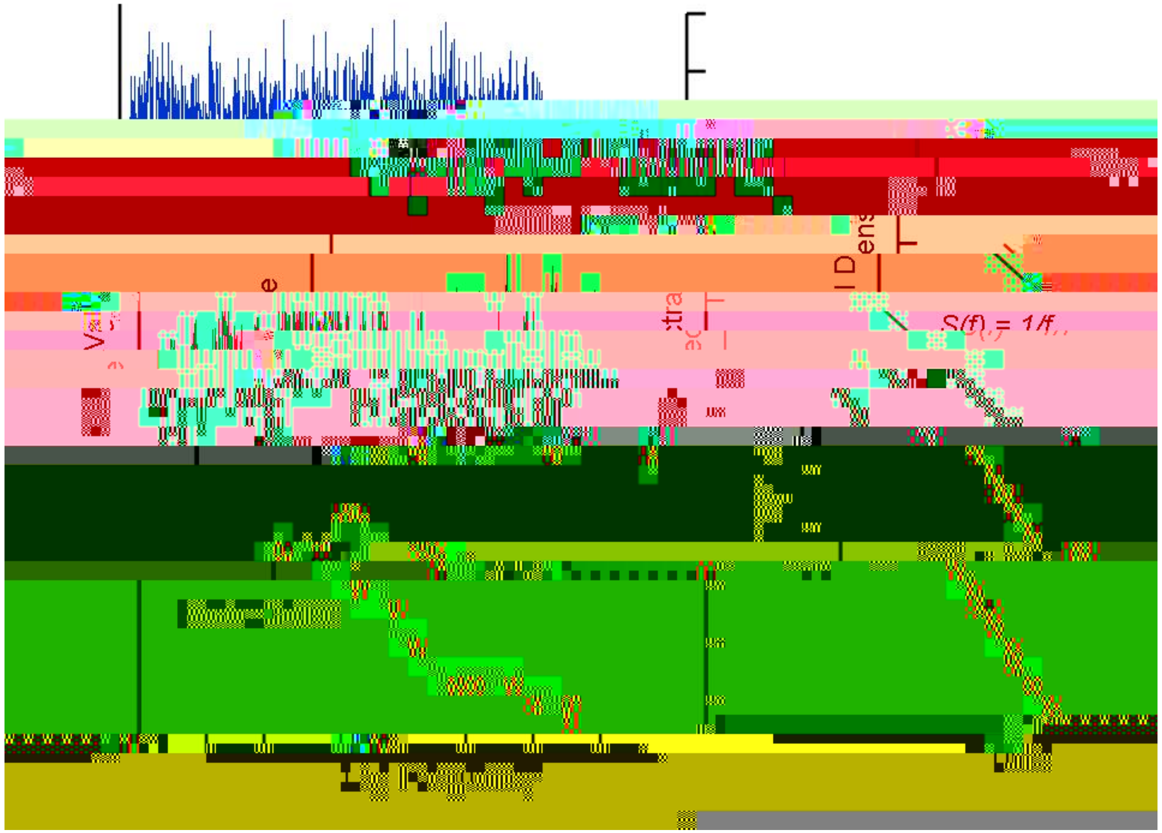
- **GFF** on the two-dimensional disk: $\mathbb{D} = |z| < L$ where $z = x + iy$. The Green function is given by $G(z_1, z_2) = -\frac{1}{2\pi}\ln\frac{L|z_1 - z_2|}{L^2 - z_1 z_2}$. In particular, for any two points $|z_{1,2}| \ll L$ (i.e. well inside the disk) we recover the **full-plane** formula

$$G(z_1, z_2) = -\frac{1}{2\pi}\ln\frac{|z_1 - z_2|}{L}\ ,\quad P[V(\mathbf{x})] \propto \exp -\frac{1}{2}\int [\nabla V(\mathbf{x})]^2\, d^2\mathbf{x}$$

- Using the full-plane logarithmic **GFF** we can construct various one-dimensional Gaussian random processes with **logarithmic** correlations. In particular, sampling the values of such **GFF** along a circle of unit radius with coordinates $z = e^{it},\ t \in [0, 2\pi)$ we get a Gaussian process with mean zero and the covariance

$$\langle V(t_1)V(t_2)\rangle = -\frac{1}{2\pi}\ln|e^{it_1} - e^{it_2}|$$

Such a process turns out to be equivalent to the random Fourier series of the form $V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[v_n e^{int} + \overline{v}_n e^{-int} \right]$, where $v_n; \overline{v}_n$ are i.i.d. complex Gaussian variables with variance $\langle v_n \overline{v}_n \rangle = 1$. As the power associated with a given Fourier harmonic with index n decays like $1/n$ such signals are known as $1/f$ noises believed to be ubiquitous in Nature.



Problem:

- Given an instance of the full-plane $2D$ **Gaussian free field**:

$$P[V(\mathbf{x})] \propto \exp -\frac{1}{8\pi g^2}\int [\nabla V(\mathbf{x})]^2 \, d^2\mathbf{x}$$

characterized by the covariance

$$\langle V(\mathbf{x}_1)V(\mathbf{x}_2)\rangle = -2g^2 \ln|\mathbf{x}_1 - \mathbf{x}_2|$$

we wish to understand the statistics of its **minima/maxima** along various curves in the plane, and ultimately in various planar domains.

- The problem turns out to be intimately connected to the mechanism of **freezing transitions** in disordered systems theory (Random Energy Models, Dirac fermions in random magnetic field). It has also interesting relations to **Liouville Quantum Gravity** & conformal field theory, to **multifractal** random measures, $1/f$ **noises**, and processes arising in turbulence and mathematical finance, as well as to various aspects of **Random Matrix Theory**.

Idea of the method: We concentrate on considering samples of the full-plane Gaussian Free Field (2d**GFF**) along **planar curves** C parametrised by $\mathbf{x}(t) = (x(t), y(t))$ with real $t \in [a, b]$.
Given a measure $d\mu_\rho(t) = \rho(t)\,dt$, we consider the integral

$$Z_\beta = \epsilon^{\beta^2 g^2} \int_a^b e^{-\beta V_\epsilon(\mathbf{x}(t))} d\mu_\rho(t), \quad \beta > 0$$

where $V_\epsilon(\mathbf{x})$ is the regularized version of the 2d**GFF** with a short scale cutoff $\epsilon \ll 1$, i.e. zero mean and the covariance

$$\langle V_\epsilon(\mathbf{x}) V_\epsilon(\mathbf{x}') \rangle = -2g^2 \ln |\mathbf{x} - \mathbf{x}'|_\epsilon = \begin{cases} -2g^2 \ln |\mathbf{x} - \mathbf{x}'|, & |\mathbf{x} - \mathbf{x}'| > \epsilon \\ 2g^2 \ln(1/\epsilon), & |\mathbf{x} - \mathbf{x}'| < \epsilon \end{cases}$$

The integral is to be interpreted as the **partition function** of the associated **Random Energy Model** at the temperature $T = \beta^{-1}$. This is to be studied in the limit $\epsilon \to 0$.

Guiding example: CIRCULAR LOGARITHMIC MODEL:

Let the contour C be the unit circle: $x(t) = \cos t; y(t) = \sin t$, with $t \in [0; 2\pi)$. Sample the 2d Gaussian Free Field at M equidistant points along the circle with $t_k = \frac{2\pi}{M}(k - 1); \; k = 1; \ldots; M$

As the distance $|\mathbf{x}_1 - \mathbf{x}_2|$ between a pair of points is simply $2|\sin \frac{t_1 - t_2}{2}|$, we deal with the collection of M normally distributed variables with covariances

$$\langle V_k V_m \rangle = -2g^2 \ln |2 \sin \; 2\pi$$

Observation: The positive integer moments $\langle Z^n(\beta)\rangle;\ n=1,2,\dots$ of the partition function $Z(\beta)=\sum_{i=1}^{M} e^{i\beta V_i}$ for the circular logarithmic model in the high-temperature phase $\gamma=\beta^2 g^2<1$ turn out to be given in the thermodynamic limit $M\gg\infty$ by

$$\langle Z^n_{circ}(\beta)\rangle = \begin{cases} M^{1+\gamma n^2}\,O(1) & n>1/\gamma \\ M^{(1+\gamma)n}\,D_n(\gamma) & n<1/\gamma \end{cases}$$

where $D_n(\gamma)$ is the **Dyson-Morris** Integral

$$D_n(\gamma)=\frac{1}{(2\pi)^n}\int_0^{2\pi} d\theta_1 \dots \int_0^{2\pi} d\theta_n$$

Outcome of the analysis:

The probability density $P(Z)$ of the partition function $Z_{circ}(\beta) \equiv Z$ in the high-temperature phase $\gamma = \beta^2 g^2 < 1$ consists of two pieces. The **"body"** of the distribution is given by:

$$P(Z) = \frac{1}{\gamma}\frac{1}{Z}\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}} e^{-\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}}}; \quad Z \ll M^2$$

which has a pronounced maximum at $Z \sim Z_e = \frac{M^{1+\gamma}}{\Gamma(1-\gamma)} \ll M^2$, and the powerlaw decay at $Z_e \ll Z \ll M^2$.

At $Z \gg M^2$ the above expression is replaced by the **lognormal tail**:

$$P(Z) = \frac{M}{\sqrt{4\pi\gamma \ln M}}\frac{1}{Z} f\left(\frac{1}{2}\frac{\ln Z}{\ln M}\right) e^{-\frac{1}{4\ln M\,\gamma}\ln^2 Z} \quad \text{where } f(x) \sim O(1) \text{ for } x \sim O(1)$$

Now we define $z = Z/Z_e$, put the coupling constant $g = 1$ and consider the generating function

$$g_\beta(x) = \left\langle \exp(-e^{\beta x} z)\right\rangle_{M \gg 1}; \quad \beta = 1/T$$

Freezing scenario: In the high-temperature phase $\beta < \beta_c = 1$ the generating function $g_\beta(x)$ can be found explicitly and turned out to satisfy a remarkable **duality relation**:

$$g_\beta(x) = \int_0^\infty dt \exp\left\{ -t - e^{\beta x} t^{-\beta^{-2}} \right\} \Rightarrow g_\beta(x) = g_{\frac{1}{\beta}}(x) \, .$$

This however does not allow to continue to $\beta > \beta_c$ regime. The phase transition at $\beta = \beta_c$ is believed to be described by the following **freezing scenario**: $g_\beta(x)$ **freezes** to the **temperature independent** profile $g_{\beta_c}(x)$ in the "glassy" phase $T \le T_c$. The scenario is supported by

(i) a heuristic **real-space renormalization group arguments** for the logarithmic models (**Carpentier, Le Doussal ’01**) revealing an analogy to the **travelling wave** analysis of polymers on disordered trees (**Derrida, Spohn 1989**)

(ii) duality which implies

$$\partial_\beta g_\beta(x)|_{\beta=\beta_c} = 0 \quad ; \text{ for all } x$$

showing that the "temperature flow" of this function vanishes at the critical point $\beta = \beta_c = 1$

(iii) our **numerics**.

Assuming validity of such scenario for the problem in hand, one finds the frozen profile for the circular model:

$$g_{\beta_c}^{circ}(x) = 2e^{x/2}K_1(2e^{x/2})$$

where $K_1(z)$ is the Macdonald function. This allows to reconstruct the **distribution of the free energy** $f = -\beta^{-1}\ln z$ for any $T < T_c$. The corresponding formula takes a form of an infinite series:

$$P^{CLM}_{\beta>\beta_c}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isf}\,\frac{1}{\Gamma\left(1+\frac{is}{\beta}\right)}\,\Gamma^2\left(1+\frac{is}{\beta_c}\right) ds$$

$$= -\frac{d}{df}\left[1+\sum_{n=1}^{\infty}\frac{e^{n\beta_c f}}{n!(n-1)!\,\Gamma\left(1-n\frac{\beta_c}{\beta}\right)}\left(\beta_c f+\frac{1}{n}-2\psi(n+1)+\frac{\beta_c}{\beta}\psi\left(1-n\frac{\beta_c}{\beta}\right)\right)\right]$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In the zero temperature limit $\beta \to \infty$ the free energy distribution yields the **extreme value probability density.**

The minimum of the random potential is simply given by $V_{min} = -\lim_{T\to 0} f = const + x$, with known *const* and the probability density of x related to the frozen profile $g_{\bar{c}}(x)$ by

$$p(x) = -g'_{\bar{c}}(x) = -\frac{d}{dx}\left[2e^{x/2}K_1(2e^{x/2})\right] \tag{1}$$

This is different from **Gumbel** distribution $p_{Gum}(x) = -\frac{d}{dx}\left\{\exp - Be^{Ax}\right\}$.

From circles to intervals:

For integers $n = 1; 2; \ldots$, a well defined and universal $^{2} \to 0$ limits exist for the moments of the partition function

$$\left\langle Z^{n}_{[0;1]} \right\rangle = \int_0^1 \cdots \int_0^1 \prod$$

dt

$\frac{1}{i}\, Q$

$G(x)$ for $[0;1]$ **interval without weight:** $a = b = 0$**:** Let $\bar{\ } < \bar{\ }_c = 1$. Introduce the

$\frac{i}{2} e^{ixt}$

Fateev, Zamolodchikov, Zamolodchikov 2000

1. Not

In this way we arrive to

$$M_\beta(s) = A_\beta 2^{(s-1)(2+\beta^2(2s+1))}\pi^{1-s}$$

$$\times \frac{-(1+\beta^2(s-1))\Gamma_\beta(\frac{\beta}{2}+\frac{1}{\beta}+\beta s)\Gamma_\beta(\frac{3}{2\beta}+\beta s)\Gamma_\beta(\frac{\beta}{2}+\frac{3}{2\beta}+\beta s)}{\Gamma_\beta(\beta+\frac{2}{\beta}+\beta s)\Gamma_\beta(\frac{1}{\beta}+\beta s)^2}$$

with $A_\beta = \frac{\Gamma_\beta(\frac{1}{\beta}+\beta)^2 \Gamma(2\beta+\frac{2}{\beta})}{\Gamma_\beta(\frac{3\beta}{2}+\frac{1}{\beta})\Gamma_\beta(\frac{3}{2\beta}+\beta)\Gamma_\beta(\frac{3\beta}{2}+\frac{3}{2\beta})}$. To guarantee that we have found the correct continuation, we have checked

(i) positivity: $M(s)$ given above is finite and positive on the interval $s \in [0; +\infty[$ that is all real moments $n = 1 - s < 1$ exist.

(ii) convexity: on this interval $\partial_s^2 \ln M(s) > 0$.

(iii) For **integer** values of s gives back known positive/negative moments.

We can use the above expression to extend the **duality relation**:

$$g_\beta(x) = g_{\frac{1}{\beta}}(x):$$

to the case of the interval $[0; 1]$.

Under the **freezing hypothesis** we extract the frozen profile $g_{-c}(x)$. For the general case the expression can be obtained as expansion in powers of e^x for $x \to -\infty$. For example, for $a = b = 0$

$$g_{-c}(x \to -\infty) = 1 + (x + A')e^x + (A + By + Cx^2 + \frac{1}{6}x^3)e^{2x} + \dots \tag{2}$$

with $A' = 2\gamma_E + \ln(2\pi) - 1$ and $C = -0.253846$, $B = 1.25388$, $A = -5.09728$. For the special case $a = b = -1/2$ we obtain the closed form expression:

$$g_{-c}(x) = \frac{\pi}{4}\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}} e^{-\frac{t^2}{2} - 2\sqrt{\ln 2}\,t} \int_{e^x}^{\infty} \left(1 - \frac{e^x}{u}\right) e^{-\sqrt{\pi u/2}\, e^{-\sqrt{\ln 2}\,t}}\, du$$

Although these expressions are different from the circle case, the universal **Carpentier-Le Doussal tail** for the probability density of **extreme values**

$$p(x \to -\infty) = -g'_{-c}(x \to -\infty) \approx -x e^x$$

is shared by all these distributions. It has its origin in the characteristic tail of the partition function density $P(z \gg 1) \propto 1/z^2$ developed at criticality, with the first moment $< z >$ becoming **infinite**.

Conclusions & Discussions:

- Using the methods of statistical mechanics we were able to extract the explicit expressions for distributions of extrema of the Gaussian Free Field sampled along **(i)** circles of unit radius and **(ii)** intervals of unit length. The distributions are manifestly **non-Gumbel** and show **universal backward tail**. The results are expected to describe extreme value statistics for $1/f$ signals, and in this way could be

Our method was based on a few assumptions, most importantly **(i)** freezing scenario for REM-type models, and **(ii)** ability to continue Selberg integrals away from positive integers to the complex plane (can be put on the rigorous basis by a method developed recently in **D. Ostrovsky Comm. Math. Phys. 288 (2009) 287-310**)

It remains a challenge:

- to verify/justify the freezing scenario
- to understand universality of the results for other 1*d* curves
- to access extreme value statistics of GFF in 2D domains.

Related work in progress:

Statistics of velocities in **decaying Burgers turbulence** with correlated initial conditions $< v(x)v(x') > \sim |x - x'|^{-2}$.