Outline

- **Critical random matrix ensembles**

- **Perturbation series for fractal dimensions**

  - **Strong multifractality**
  - **Weak multifractality**

- **Conjecture:** $\chi = 1 - D_1/d$

- **Summary**

Well accepted conjectures

- **Berry, Tabor (1977):**

  Integrable systems $\Longrightarrow$ Poisson statistics

$$(\Delta+E)\Psi=0$$

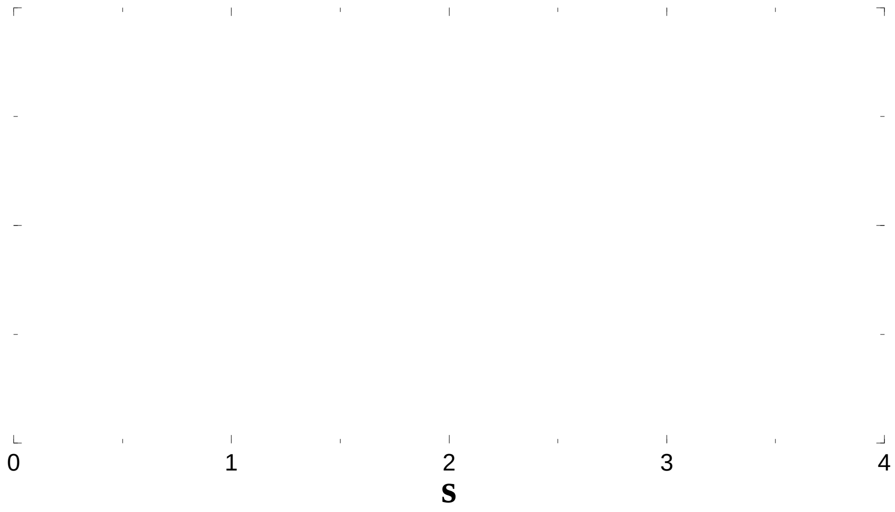
3-d Anderson model at metal-insulator transition

3-d Anderson model

$$H = \sum_{i} \varepsilon_i a_i a_i - \sum_{j = \text{adjacent to } i} a_j a_i$$

ε_i = i.i.d. random variables between $-W/2$ and $W/2$

Spectral characteristics of 3-d Anderson model at metal-insulator transition



Characteristic features of intermediate statistics

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Random matrix models of intermediate statistics

$$M_{ij} = \varepsilon_j \delta_{ij} + V(i-j)$$

Typically:

$$V(i-j) \Box \frac{g}{|i-j|^{\alpha}}$$

ε

Critical power law banded random matrices

(Mirlin et al (1996))

$N \times N$ Hermitian matrices whose elements, H_{ij}, are i.i.d. Gaussian variables (real for $\beta = 1$ and complex for $\beta = 2$) with zero mean $\langle H_{ij} \rangle = 0$ and the variance $\langle |H_{ii}|^2 \rangle = 1/\beta$ and

$$\langle |H_{ij}|^2 \rangle = \left(1 + \frac{(i-j)^2}{b^2}\right)^{-1} \quad \text{for } i \neq j$$

Perturbation series: (Mirlin, Evers (2000))

- $b \gg 1$: $D_q = 1 - q/(2\pi\beta b)$, $\chi = 1/(2\pi\beta b)$.

$$\boxed{D_q = 1 - q\chi}$$

- $b \ll 1$: ($c_\beta = 1$ for $\beta = 1$, $c_\beta = \pi/\sqrt{8}$ for $\beta = 2$)

$$D_q = 4bc_\beta \frac{\Gamma(q - 1/2)}{\sqrt{\pi}\Gamma(q)} , \quad \chi = 1 - 4bc_\beta$$

$$\boxed{D_q = \frac{\Gamma(q-1/2)}{\sqrt{\pi}\Gamma(q)}(1 - \chi)}, \qquad \boxed{D_1 = 1 - \chi}$$

Absence of universality for spectral statistics

Ruijsennars-Schneider ensemble

(E.B., Schmit, Giraud (2009))

- **Ruijsenaars - Schneider classical integrable model**

$$H(p,q) = \sum_{j=1}^{N} \cos(p_j) \prod_{k=j} \left(1 - \frac{\sin^2 \pi a}{\sin^2 \frac{\mu}{2}(q_j - q_k)}\right)^{1/2}$$

- **Ruijsenaars - Schneider ensemble of random matrices**

$N \times N$ unitary matrix related with the Lax matrix of this model

$$M_{kp} = e^{i\Phi_k} \qquad 1 - e$$

$1 < a < 2$

$$p(s) \equiv p(1, s) = 0 \text{ for } s > a$$

$$p(2, s) = 0 \text{ for } s < a \text{ and } s > 2a$$

$$p(3, s) = 0 \text{ for } s < a \text{ and } s > 3a$$

$a = 4/3$

$$p(s) = \frac{81}{64}s^2, \; 0 < s < a, \quad p(2, s) = \left(-\frac{3}{2} + \frac{27}{16}s - \frac{81}{512}s^3\right) e^{3s/}$$

$2 < a < 3$

Tedious calculations and complicated expressions

$$p(s) \equiv p(1, s) = 0 \text{ for } s > a$$

$$p(2, s) = 0 \text{ for } s > a$$

$$p(3, s) = 0 \text{ for } s < a \text{ and } s > 2a$$

Fractal dimensions

Fractal dimensions are not yet accessible for analytical calculations

Perturbation series = the only analytical way to them

The Ruijsenaars-Schneider ensemble:

$$M_{mn} = e^{i\Phi_m} \frac{1 - e^{2\pi i a}}{N\left(1 - e^{2\pi i (m-n+a)/N}\right)}$$

Perturbation series are possible around all integer points $a = k$.

$a = k + \epsilon$

$$M_{mn} = M^{(0)}_{mn}\left(1 + \frac{\pi i (N-1)}{N}\epsilon\right) + \epsilon M^{(1)}_{mn} + O(\epsilon^2)$$

where

$$M^{(0)}_{mn} = e^{i\Phi_m}\delta_{n,m+k}$$

$$M^{(1)}_{mn} = e^{i\Phi_m}(1 - \delta_{n,m+k})\frac{\pi e^{-\pi i(m-n+k)/N}}{N \sin(\pi(m k}$$

Perturbation series for strong multifractality: $|D_q|$ □ 1

a □ 1 $\longrightarrow$ $M^{(0)}_{mn}$ is diagonal $\longrightarrow$ degenerate perturbation series

- Unperturbed eigenfunctions $\Psi^{(0)}_j(\alpha) = \delta_{j\,\alpha}$
- Unperturbed eigenvalues $\lambda_\alpha = e^{i\Phi_\alpha}$

The first order = the contributions of 2 × 2 sub-matrices

$$\begin{pmatrix} M_{mm} & M_{mn} \\ M_{nm} & M_{nn} \end{pmatrix} \equiv \begin{pmatrix} e^{i\Phi_m} & e^{i\Phi_m} h \\ -e^{i\Phi_n} h & e^{i\Phi_n} \end{pmatrix}$$

Mean moments of eigenfunctions

$$I_q = \frac{1}{N\rho(E)} \sum_{j,\alpha=1}^{N} |\Psi_j(\alpha)|^{2q} \delta(E - E_\alpha)\rangle .$$

$\rho(E)$ = the total mean eigenvalue density. For RSE: $\rho(E) = 1/2\pi$

Fractal dimensions

where

$$J(q) = \int_{-}\left[\frac{1}{(1+e^{2t})^q} + \frac{1}{(1+e^{-2t})^q} - 1\right]\cosh(t)dt = -\frac{\sqrt{\pi}\,\Gamma\left(q-\frac{1}{2}\right)}{\Gamma(q-1)}$$

One gets

$$\sum_{j=1}^{N-1}\frac{\pi}{N\sin(\pi j/N)} = 2\ln N + 2(\gamma + \ln 2 - \ln\pi)$$

Finally when $N \to \infty$

$$I_q \longrightarrow -2a\frac{\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q-1)}\ln N$$

By definition $I_q \longrightarrow N^{-(q-1)D_q}$

Perturbation series for week multifractality:

$$|1 - D_q| \,\square\, 1$$

When a = k + □ and k □ 0 the unperturbed matrix

$$M^{(0)}_{mn} = e^{i\Phi_m}\delta_{n,\,m+k}$$

is the shift matrix and its eigenfunctions are extended

The case k = 1

Eigenvalues $\lambda(\alpha)$ and eigenfunctions $\Psi^{(0)}_n(\alpha)$ of $M^{(0)}_{mn}$ are

$$\lambda(\alpha) = e^{i\bar{\Phi} + 2\pi i\alpha/N}, \qquad \Psi^{(0)}_n(\alpha) = \frac{1}{\sqrt{N}}e^{iS_n(\alpha)},$$

$$S_n(\alpha) = \frac{2\pi}{N}\alpha(n-1) - \sum_{j=1}^{n-1}(\Phi_j - \bar{\Phi}), \quad \bar{\Phi} = \sum_{j=1}^{N}\Phi_j \;\; \alpha$$

The first order in $\epsilon = a - 1$

$$C_{\alpha\beta} = \square \frac{\sum_{mn} \Psi_m^{(0)}(\beta) M_{mn}^{(1)} \Psi_n^{(0)}(\alpha)}{\lambda(\alpha) - \lambda(\beta)}$$

At the leading order in □

$$\left\langle \sum_{n=1}^{N} |\Psi_n(\alpha)|^{2q} \right\rangle = N^{1-q}\left[1 + \frac{q(q-1)}{2} W(\alpha)\right],$$

$$W(\alpha) = \frac{1}{N} \sum_{n=1}^{N} \left\langle \left[\sum_{\beta} e^{iS_n(\beta) - iS_n(\alpha)} C_{\alpha\beta} + c.c. \right]^2 \right\rangle.$$

Direct (but tedious) calculations show strong cancellations and

$$W(\alpha) = \square^2 \frac{\pi^2}{N^3} \sum_{\beta=1}^{N-1} \sum_{n=1}^{N-1} \frac{\sin^2(\pi\beta n/N)}{\sin^2(\pi n/N)\sin^2(\pi\beta/N)}.$$

When N N

Fractal dimensions for RSE

The remaining sum over n can be transformed into an integral over y and when $N \to \infty$

$$W(\alpha) \longrightarrow 2\square^2 \ln N + O(1).$$

Combining all terms together one finds

$$D_q = 1 - q(1 - a)^2.$$

For $k \geq 2$ calculations are more tedious but one can prove that

$$D_q = 1 - q\frac{(a - k)^2}{k^2} \text{ when } |a - k| \square 1$$

For comparison when

Spectral compressibility for RSE

- **0 < a < 1**

$$\chi = (1 - a)^2.$$

$$\chi \longrightarrow 1 - 2a, \quad |a| \ \square \ 1$$

- **1 < a < 2**

$$\chi = \left(\frac{a^2}{4} - \frac{4a(1-a)z^2 + a^2 \sinh^2 z}{(2z - \sinh 2z)^2} \sinh^2 z \right) \frac{\sinh^2 z}{z^2}$$

where z is the solution of

$$a = \frac{2z^2 - z \sinh 2z}{z^2 + \sinh^2 z - z \sinh 2z}$$

z is real when $1 < a < 4/3$

z is imaginary when $4/3 < a < 1$

For $a = 4/3$, $\chi = 1/9$

- **2 < a < 3**

$$\chi = \frac{1}{a(\sin^2 z + z^2 - z\sin 2z)^2}\Big[(a-3)^2(a-2)z^2 - 6(a-2)z^2\sin^2 z$$

$$-(a-3)(a-1)(2a-5)z^3\sin 2z + 2(a-2)(\cos 2z + 2)(a-1)(a-2)z^2\sin^2 z$$

$$-2a(a-2)(2a-3)z\cos z\sin^3 z + a(a-1)^2\sin^4 z\Big]$$

where

$$x = \frac{a\sin^2 z + (a-2)z^2 + (1-a)z\sin 2z}{(a-1)\sin^2 z + (a-3)z^2 + (2-a)z\sin 2z}$$

and

$$\frac{e^x}{x} = \frac{\sin z}{z}e^{z/\tan z}$$

From exact expressions it follows

$$\chi \longrightarrow \begin{cases} 1 - 2a & |a| \;\square\; 1 \\ \dfrac{(a-k)^2}{k^2} & |a-k| \;\square\; 1 \text{ and } |k| \ge 1 \end{cases}$$

1a

Fractal dimensions for CrBRME and RSE

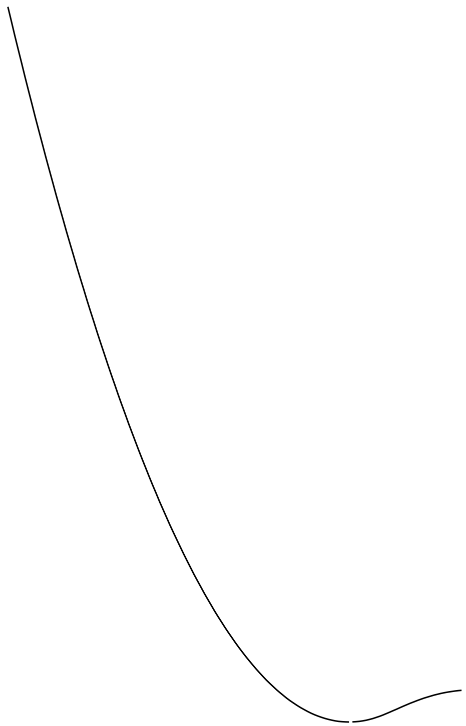
CrBRME	RSE
Weak multifractality	
$1/b$ □ 1	$\lvert a-k\rvert$ □ 1
$D_q = 1 - q\frac{1a-}{2\pi\beta b}$	$D_q = 1 - q\frac{(a-k)^2}{k^2}$
$\chi = \frac{1}{2\pi\beta b}$	$\chi = \frac{(a-k)^2}{k^2}$
Strong multifractality	
b □ 1	$\lvert a\rvert$ □ 1
$D_q = 4bc_\beta$	

Conjecture: $\chi = 1 - D_1/d$, **(E.B. and Giraud (2010))**

Wave function entropy (information dimension):

$$-\left\langle \sum_n |\Psi_n(\alpha)|^2 \ln |\Psi_n(\alpha)|^2 \right\rangle \longrightarrow D_1 \ln N$$

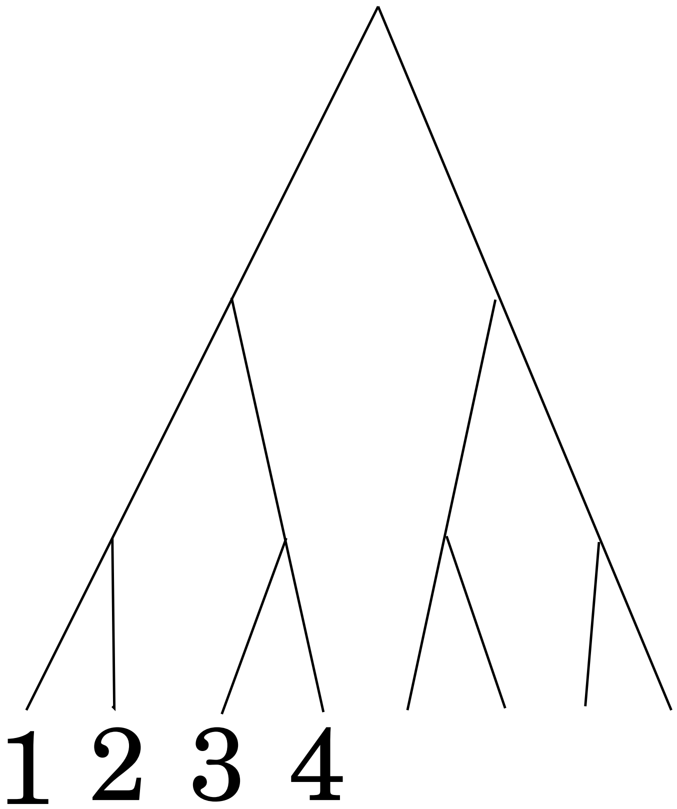
Chalker, Kravtsov, Lerner (1996): $\chi = 1/2 - D_2/2d$



Critical ultrametric matrices

(Fyodorov, Ossipov, Rogriguez (2009))

$2^K \times 2^K$ Hermitian matrices with independent Gaussian variables with zero mean and $\langle |H_{nn}|^2 \rangle = W^2$. $\langle |H_{mn}|^2 \rangle = 2^{2-d_{mn}} J^2$, d_{mn} = the ultrametric distance between m and n along the binary tree



	1	1/2	1/2	1/4	1/4	1/4	1/4
1		1/2	1/2	1/4	1/4	1/4	1/4
1/2	1/2		1	1/4	1/4	1/4	1/4
1/2	1/2	1		1/4	1/4	1/4	1/4
1/4	1/4	1/4	1/4		1	1/2	1/2
1/4	1/4	1/4	1/4	1		1/2	1/2
1/4	1/4	1/4	1/4	1/2	1/2		1
1/4	1/4	1/4	1/4	1/2	1/2	1	

Higher dimensional conjecture: $\boxed{\chi = 1 - D_1/d}$

Standard two-dimensional critical model:

MIT in the quantum Hall effect

via the Chalker-Coddington network model

(Evers et al. (2008)) $\longrightarrow$ $D_1 = 1$

Summary

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Compressibility for ultrametric ensemble

By definition: $\boxed{\chi = 1 + \lim_{L} \quad \lim_{N} \quad F_{L,N}}$,

$$F_{L,N} = \frac{1}{\bar{\rho}} \int_{-L/\bar{\rho}}^{L/\bar{\rho}} [R_2(E + s/2, E - s/2) - \bar{\rho}^2] ds.$$

Here $R_2(E_1, E_2)$ is the two-point correlation function

$$R_2(E_1, E_2) = \left\langle \sum_{m=n}^{N} \delta(E_1 - \lambda_m)\delta(E_2 - \lambda_n) \right\rangle,$$

$\bar{\rho} = N\rho(E)$ is the mean density.

In the first order of perturbation series it is sufficient to consider

2 × 2 sub-matrix

$$\begin{pmatrix} H_{mm} & H_{mn} \\ H_{nm} & H_{nn} \end{pmatrix}$$

$$R_2(E + s/2, E - s/2) =$$

$$= \sum_{n \neq m} \delta(E + s/2 - \xi - \sqrt{}$$

$$F_{L,N} \approx 2\rho \left\langle \sum_{i=i_0}^{K-1} 2^i \sqrt{\left(\frac{L}{\rho N}\right)^2 - 4\left(\frac{J|z|}{2^i}\right)^2} \quad - 2L$$

with i_0 such that $L/(2\rho N) \approx J|z|/2^{i_0}$.

$$\sum_{i=i_0}^{K-1} 2^i = 2^K - 2^{i_0} = N - \frac{2J|z|N}{L\rho}$$

Therefore

$$F_{L,N} \approx 2\rho \left\langle \sum_{i=i_0}^{K-1} 2^i \left[\sqrt{\left(\frac{L}{\rho N}\right)^2 - 4\left(\frac{J|z|}{2^i}\right)^2} - \frac{L}{\rho N}\right] - 2J|z| \right\rangle$$

Change i to $2J|z|/2^i = L/(x\rho N)$. Then

$$F(L,N) \approx 4\rho \frac{J}{\ln 2} \left\langle |z| \left[\int_1^{x_m} \left(\sqrt{1 - 1/x^2} - 1\right) dx - 1 \right] \right\rangle$$

where $x_m = L/(4\rho J|z|)$. In the limit $L \to \infty$, $x_m \to \infty$.
$\int_1 \left(\sqrt{1 - 1/x^2} - 1\right) dx = 1 - \pi/2$, $\langle |z| \rangle = \sqrt{\pi}/2$, and $\rho(0) = 1/\sqrt{2\pi}W$:

$$\chi = 1 - \frac{\pi J}{\sqrt{2} \ln 2\, W}$$