Universality in the two matrix model with one quartic potential

Maurice Duits
Caltech

December 18, 2010

VI Brunel Workshop on Random Matrix Theory

Eigenvalue density

The tansformed functions $Q_{j;n}$ and $P_{k;n}$

- Introduce the transformed funtions

$$Q_{j;n}(x) = e^{-nV(x)} \int q_{j;n}(y) e^{-n(W(y) - xy)}\, dy$$

$$P_{k;n}(y) = e^{-nW(y)} \int p_{k;n}(x) e^{-n(V(x) - xy)}\, dx$$

- Note that we have the orthogonality relations

$$\int p_{k;n}(x) Q_{j;n}(x)\, dx = 0, \qquad j \neq k$$

$$\int P_{k;n}(y) q_{j;n}(y)\, dy = 0, \qquad j \neq k$$

- Let

$$h_{k;n}^2 = \iint p_{k;n}(x) q_{k;n}(y) e^{-n(V(x) + W(y) - xy)}\, dx\, dy$$

Four kernels

Define kernels by

$$K_{11}(x_1;x_2) = \sum_{k=0}^{n-1} \frac{1}{h_{k;n}^2} p_{k;n}(x_1) Q_{k;n}(x_2);$$

$$K_{22}(y_1;y_2) = \sum_{k=0}^{n-1} \frac{1}{h_{k;n}^2} P_{k;n}(y_1) q_{k;n}(y_2)$$

$$K_{12}(x;y) = \sum_{k=0}^{n-1} \frac{1}{h_{k;n}^2} p_{k;n}(x) q_{k;n}(y)$$

$$K_{21}(y;x) = \sum_{k=0}^{n-1} \frac{1}{h_{k;n}^2} P_{k;n}(y) Q_{k;n}(x) - e^{-n(V(x)+W(y)-\tau xy)}$$

Eynard-Metha Theorem

- Denote the eigenvalues of M_1 by $x_1; : : : ; x_n$ and of M_2 by $y_1; : : : ; y_n$. The probability density function can be written as

$$P(x_1; : : : ; x_n; y_1; : : : ; y_n) = \frac{1}{n!^2} \det \quad K$$

Averaging over M_2

- When averaged over M_2 we see that the eigenvalues of M_1 describe a determinantal point process with kernel K_{11}.

$$\underbrace{\int \cdots \int}_{n-k \text{ times}} P(x_1, \ldots, x_n)\,\mathrm{d}x_{k+1} \cdots \mathrm{d}x_n = \frac{(n-k)!}{n!} \det \left[K_{11}(x_i, x_j) \right]_{i,j=1}^{k}$$

- This is a particular example of a so-called biorthogonal ensemble.

Asymptotic analysis

Question: Find a full asymptotic description of the biorthogonal polynomials and the associated kernels.

- There exist several Riemann-Hilbert characterizations of the biorthogonal polynomials
  Ercolani-Mclaughlin '01, Kapaev '03, Bertola-Eynard-Harnad '03, Kuijlaars-McLaughlin '05
- Except for the special in which both V and W are quadratic Ercolani-McLaughlin '01, a steepest descent analysis turns out to be complicated.

Multiple Orthogonality

- The main idea in Kuijlaars-McLaughlin '05 is to interpret the polynomials as multiple orthogonal polynomials.

- Define the weight function w_j by

$$w_j(x) = e^{-nV(x)} \int_{\mathbb{R}} y^j e^{-n(W(y) - xy)} \, dy, \qquad j = 0, 1, \ldots, d-2.$$

  where $d = \mathrm{degree}(W)$.

- The polynomials $p_{k,n}$ are multiple orthogonal polynomials of type II with respect to the weights w_j on $\mathbb{R}$. For $p_{n,n}$ this means that

$$\int_{\mathbb{R}} p_{n,n}(x) x^l w_j(x) \, dx = 0, \quad l = 0, \ldots, n_j - 1, \quad j = 0, 1, \ldots, d-2,$$

  where n_j is the integer part of $(n + d - 2 - j)/(d-1)$.

- For multiple orthogonal polynomials a Riemann-Hilbert characterization is known Van Assche-Geronimo-Kuijlaars '01.
- We seek for a $d \times d$ matrix valued function Y such that

$$\begin{cases} Y \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_0(x) & \cdots & w_{d-2}(x) \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad x \in \mathbb{R} \\ Y(z) = (I + O(1/z)) \operatorname{diag}(z^n, z^{-n_0}, \ldots, z^{-n_{d-2}}), \quad z \to \infty \end{cases}$$

where n_j is the integer part of $(n + d - 2 - j)/(d - 1)$.

The solution of the Riemann-Hilbert problem

- The solution exists and is unique. Moreover

$$Y_{11}(z) = p_{n,n}(z)$$

Van Assche-Geronimo-Kuijlaars '01

- Also the kernel $K_{11}^{(n)}$ can be expressed in Y

$$K_{11}^{(n)}(x,y) = \frac{1}{2\,\mathrm{i}(x-y)} \begin{pmatrix} 0 & w_0(y) & \quad & w_2(y) \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Daems-Kuijlaars '04

- A steepest descent analysis for the RH problem in the general situation is still an important open problem!

Quartic potential

$\underline{t = 0}$

The equilibrium problem for $t = 0$

We seek to minimize the energy functional

$$I({}_1; {}_2; {}_3) = \sum_{j=1}^{3} \iint \log |x - y|^{-1} \, d {}_j(x) d {}_j(y)$$

$$- \sum_{j=1}^{2} \iint \log |x - y|^{-1} \, d {}_j(x) d {}_{j+1}(y)$$

$$+ \int \left(V(x) - \frac{3}{4} {}^{4/3} |x|^{4/3} \right) d {}_1(x)$$

among all measures $({}_1; {}_2; {}_3)$ satisfying

1. ${}_1$ is a measure on $\mathbb{R}$ with ${}_1(\mathbb{R}) = 1$
2. ${}_2$ is a measure on $i\mathbb{R}$ with ${}_2(i\mathbb{R}) = 2/3$
3. ${}_3$ is a measure on $\mathbb{R}$ with ${}_3(\mathbb{R}) = 1/3$
4. ${}_2$ with

$$d (z) = \frac{\sqrt{3} \, {}^{4/3} |z|^{1/3}}{2} |dz|$$

The minimizer

The minimizer

Theorem (D-Kuijlaars '09)

The minimizer $(\ _1; \ _2; \ _3)$ *has the following properties*

1. $_1$ *is supported on nitely many intervals* $[^r_{j=1}[a_j; b_j]$ *and there exists real analytic* h_j *such that*

$$\frac{\mathrm{d} \ _1(x)}{\mathrm{d}x} = h_j(x) \ \mathrm{p} \overline{(b_j - x)(x - a_j)}; \quad x \ 2 \ [a_j; b_j]$$

2. $_2$ *is supported on* $\mathrm{i}\mathrm{R}$ *and* $_2 = \quad$ *on* $\mathrm{i}[-c; c]$*. Moreover, there exists an analytic function* $_2$ *such that*

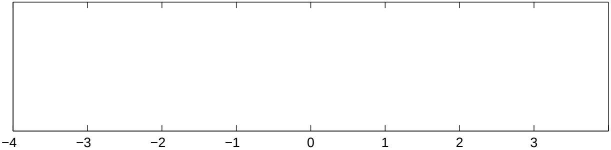
$$\mathrm{d}(\quad - \ _2)(y) = \ _2(y)|\mathrm{d}y|$$

and $_2$ *vanishes as as square root near* $y = \quad ic$*.*

3. $_3$ *is supported on* R *and there exists a function* $_3$ *which is real analytic in* $\mathrm{R} \setminus \{0\}$ *and such that*

$$\mathrm{d} \ _3(x) = \ _3(x)\mathrm{d}x$$

Example: $V(x) = x^2$=2 and $= 1$



$t \neq 0$ and V even

The equilibrium problem for $t \neq 0$ and V even

We seek to minimize the energy functional

$$I(\mu_1, \mu_2, \mu_3) = \sum_{j=1}^{3} \iint \log |x-y|^{-1} \, d\mu_j(x) d\mu_j(y)$$

$$- \sum_{j=1}^{2} \iint \log |x-y|^{-1} \, d\mu_j(x) d\mu_{j+1}(y)$$

$$+ \int V_1(x) \, d\mu_1(x) + \int V_3(x) \, d\mu_3(x)$$

among all measures (μ_1, μ_2, μ_3) satisfying

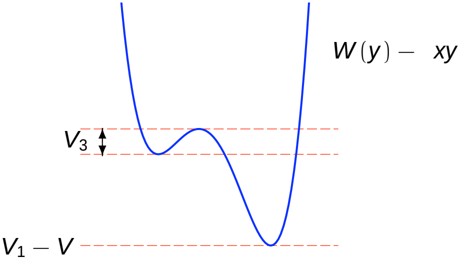
1. μ_1 is a measure on $\mathbb{R}$ with $\mu_1(\mathbb{R}) = 1$
2. μ_2 is a measure on $i\mathbb{R}$ with $\mu_2(i\mathbb{R}) = 2/3$
3. μ_3 is a measure on $\mathbb{R}$ with $\mu_3(\mathbb{R}) = 1/3$
4. $_2$

De nition of V_1

- The external eld V_1 is de ned by

$$V_1(x) = V(x) + \min_{y \, 2 \, \mathrm{R}} \left(W(y) - \ xy \right); \quad x \; 2 \; \mathrm{R}$$

- The external eld V_3 is the di erence between the other two extreme values of $W(y) - \ xy$ (viewed as a function in y).



$|x| > x \ $ and $t < 0$.

De nition of

Again we consider

$$W^0(\,!\,) - \;\; z = \;!^3 + t\,! - \;\; z = 0;$$

but now for z 2 iR. Then

$$\frac{\mathrm{d}\;\;(z)}{|\mathrm{d}z|} = -\,\mathrm{Re}\;\, !_1(z);$$

where $!_1$ is the solution of the cub iderhe solution of theof theofofi34342(of)-348 Ton

The minimizer

Theorem (D-Geudens-Kuijlaars '10, D-Kuijlaars-Mo '10)

There is unique minimizer $(\ _1;\ _2;\ _3)$ *of the energy functional* I*. Moreover, the measure* $\ _1$ *is the weak limit of the normalized zero distribution of the polynomial* $p_{n;n}$*,*

$$\frac{1}{n} \sum_{x\ :\ p_{n,n}(x)=0} \ _x \ ! \ \ _1$$

as $n\ !\ 1$*.*

Supports of the measure

Theorem (D-Kuijlaars-Mo '10)

The minimizer $(\ _1; \ _2; \ _3)$ *has the following properties*

1. $\ _1$ *is supported on nitely many intervals* $[^r$

Theorem (D-Kuijlaars-Mo '10)

Let $W(y) = y^4/4 + ty^2/2$ and V even. Let $_1$ be the rst component of the minimizer of I. Then

- *The measure $_1$ also describes the limiting mean eigenvalues density for the matrix M_1, i.e.*

$$\lim_{n \to \infty} \frac{1}{n} K_{11}^{(n)}(x; x) = \frac{\mathrm{d}\ _1(x)}{\mathrm{d}x}$$

- *Universality:*
  For x in the bulk:

$$\lim_{n \to \infty} \frac{1}{cn} K_{11} \quad x + \frac{u}{cn}; x + \frac{v}{cn} \quad = \frac{\sin \ (u - v)}{(u - v)}$$

  For x at regular endpoints: Airy kernel.

Supports of the measure

In the analysis we distinguish the cases

Case I: $0 \in S(\ _1)$, $0 \notin S(\ -\ _2)$ and $0 \in S(\ _3)$

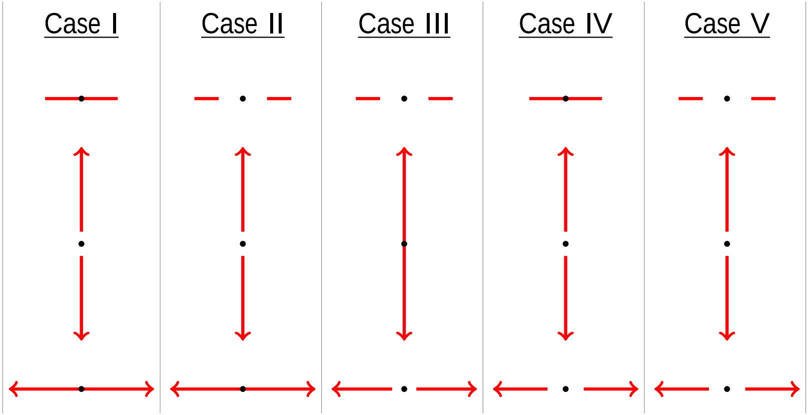
Case II: $0 \notin S(\ _1)$, $0 \notin S(\ -\ _2)$ and $0 \in S(\ _3)$

Case III: $0 \notin S(\ _1)$, $0 \in S(\ -\ _2)$ and $0 \notin S(\ _3)$

Case IV: $0 \in S(\ _1)$, $0 \notin S(\ -\ _2)$ and $0 \notin S(\ _3)$

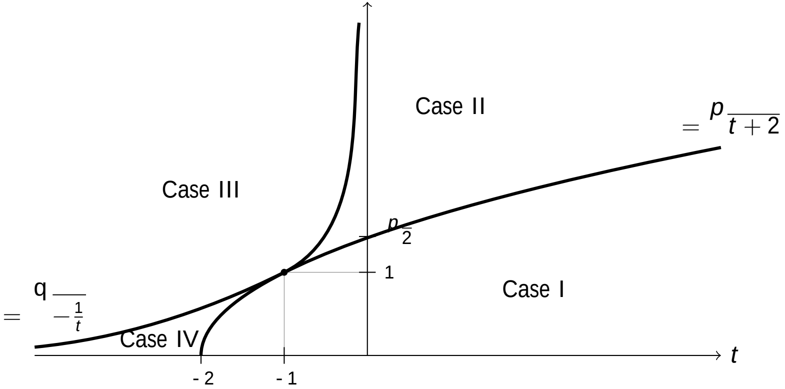
Case V: $0 \notin S(\ _1)$, $0 \notin S(\ -\ _2)$ and $0 \notin S(\ _3)$

Critical phenomena occur when going from one case to the other.



On top of each the supports $S(\;_1)$, $S(\;-\;_2)$ and $S(\;_3)$
(also the cuts of the corresponding Riemann surface)

Phase diagram for $V(x) = x^2$=2



- Case I ! Case II: Merging in $S(\ _1)$! Painleve II
- Case IV ! Case III: $S(\ _1)$ splits and $S(\ - \ _2)$ merges ! Pearcey
- Intersection point: simulataneous transition in all three measures.
  A new kernel appears D-Geudens '11?.

Derivation of the equilibrium problem

- If $V(x) = x^2/2$ then $q_{k;n}$ are orthogonal polynomials on the real line. The asymptotics of these polynomials is well-known. In particular the asymptotics for the recurrence coe cients
- The polynomials $p_{k;n}$ satisfy a ve term recurrence and the coe cients can be expressed in terms of the recurrence coe cients of the other family. So we know the asymptotic behavior of the recurrence coe cients.
- The zeros of the polynomials are the eigenvalues of the `Jacobi' matrix.
  - The zeros of the polynomials a64 -1.246 Td [(The)-342(zeros)-343(of)-343(the)

Banded Toeplitz matrices

Let $T_n(a)$ be a Toeplitz matrix

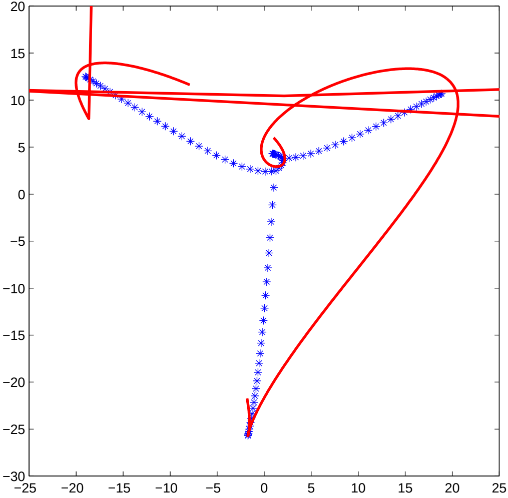
$$\big(T_n(a)\big)_{jk} = a_{j-k}, \qquad j,k = 1,\dots,n$$

for which the symbol a has only finitely many Fourier coefficients

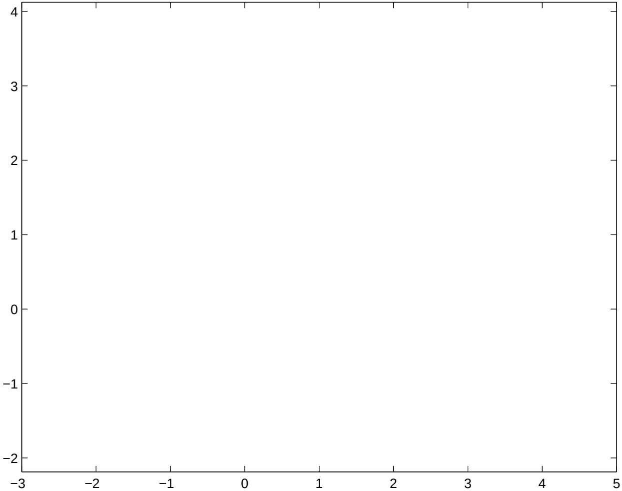
$$a(z) = \sum_{j=-q}^{p} a_j z^j, \qquad p,q > 0, \quad a_{-p}, a_q \neq 0$$

What is the limiting behavior of the spectrum $\sigma(T_n(a))$ as $n \to \infty$?

Example



Example



An associated Riemann surface

The central object to study is the algebraic equation

$$a(z) - \quad = \sum_{j=-q}^{p} a_q z^j - \quad = 0:$$

For each this equation has $p+q$ solutions which we order according to magnitude

$$0 < |z_1(\ )| \quad \ldots \quad |z_{p+q}(\ )|$$

De ne

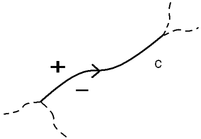
$$_k = \{ \quad | \, |z_{q+k}(\ )| = |z_{q+k+1}(\ )|\};$$

for $k = -q+1; \ldots; p-1$.

(Assumption: $\gcd\{k \mid a_k \neq 0\} = 1$)

1

- The contour $_0$ is bounded, the other are unbounded. All consist of nitely many analytic arcs



- De ne the measure $_k$ on $_k$ by

$$\mathrm{d}\ _k(\ ) = \frac{1}{}$$

1

2f 7.167 1.063 Td [(t0)]TJ/F230164 Tf 145.096 107.3254

Theorem (D-Kuijlaars '08)

The vector of measures $(_{-q+1}; \ldots ; _{p-1})$ *is the unique minimizer of the energy functional E de ned by*

$$E(_{-q+1}; \ldots _{p-1}) = \sum_{k=-q+1}^{p-1} \iint \log \frac{1}{|x-y|} \, d _k(x) \, d _k(y) - \sum_{k=-q+1}^{p-2} \iint \log \frac{1}{|x-y|} \, d _k(x) \, d _{k+1}(y)$$

where each measure $_k$ *is a measure on* $_k$ *with total mass*

$$_k(_k) = \begin{cases} \frac{q+k}{q}; & k \quad 0 \\ \frac{p-k}{p}; & k \quad 0 \end{cases}$$

Theorem (D-Kuijlaars '08)

The vector of measures $(\ _{-q+1}; :::; \ _{p-1})$ *is the unique minimizer of the energy functional E de ned by*

$$E(\ _{-q+1}; ::: \ _{p-1}) = \sum_{k=-q+1}^{p-1} \iint \log \frac{1}{|x-y|} \mathrm{d} \ _k(x) \mathrm{d} \ _k(y)$$

$$- \sum_{k=-q+1}^{p-2} \iint \log \frac{1}{|x-y|} \mathrm{d} \ _k(x) \mathrm{d} \ _{k+1}(y)$$

where each measure $\ _k$ *is a measure on* $\ _k$ *with total mass*

$$\ _k(\ _k) = \begin{array}{lll} \frac{q+k}{q}; & k & 0 \\ \frac{p-k}{p}; & k & 0 \end{array}$$

Generalization to Toeplitz matrices with rational symbols (Delvaux-D '10)

The biorthogonal polynomials were defined by the relation

$$\iint p_{k,n}(x)q_{j,n}(y)\mathrm{e}^{-n(V(x)+W(y)-xy)}\,\mathrm{d}x\mathrm{d}y=0, \qquad j\neq k$$

and we were interested in the case

$$W(y)=\frac{1}{4}y^4+\frac{1}{2}tx^2 \text{ and } V(x)=\frac{1}{2}x^2$$

and asymptotics for $p_{n,n}$ and K_{11}.

Recurrence coe cients (1)

- The orthogonal polynomials $q_{k;n}$ for $\mathrm{e}^{-n\ \ y^4/4 - \ ^2 y^2/2}$ satisfy a recurrence relation

$$yq_{k;n}(y) = q_{k+1;n}(y) + a_{k;n} q_{k-1;n}(y)$$

- Bleher and Its proved that in the limit $k, n \to \infty$ and $k/n \to \ $ we have

$$\lim_{n \to \infty;\ k/n \to \ } a_{k;n} = \frac{^2 + \sqrt{^4 + 12}}{6}; \qquad > \ ^4/4$$

and

$$\lim_{n \to \infty;\ k/n \to \ } a_{k;n} = \begin{cases} \frac{^2 - \sqrt{^4 - 4}}{}; & k \text{ even} \\ \frac{^2 + \sqrt{^4 - 4}}{2}; & k \text{ odd} \end{cases}; \qquad < \ ^4/4$$