Example of a fractal eigenfunction

Outline

1. Power-law banded matrices and ultrametric ensemble: universality of the fractal dimensions

2. Fractal dimensions: beyond the universality

3. Dynamical scaling: validity of Chalker’s ansatz

Fractal dimensions

Moments:

Metal: $d_q = d$

Insulator: $d_q = 0$

Anomalous scaling exponents: $\Delta_q \equiv (d_q - d)(q-1)$

Critical point: $\Delta_q \neq 0$

Spatial correlations : $L^{2d}\overline{|\psi^2(\mathbf{r})\psi^2(\mathbf{r}')|} \propto (|\mathbf{r}-\mathbf{r}'|/L)^{\Delta_2}$

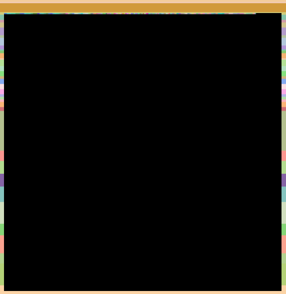
How one can calculate d_q ?

Green's functions:

$$I_{m+n} \Longleftrightarrow \overline{(G^R(\mathbf{r},\mathbf{r}))^m (G^A(\mathbf{r},\mathbf{r}))^n}$$

Power-law banded random matrices

$$\overline{|H_{ij}|^2} = \frac{1}{1+(|i-j|/b)^{2\alpha}}$$



$\alpha = 1$ critical states at all values of b

Almost diagonal matrices

Diagonal approximation: $\overline{I_q^{(0)}} = 1$

2x2 matrix approximation:

$$H(n,m) = \begin{pmatrix} H_{nn} & H_{nm} \\ H_{nm}^* & H_{mm} \end{pmatrix} \qquad \overline{I_q^{(1)}} = \frac{1}{L}\sum_{n\neq m}^{L} (\overline{I_q(n,m)} - 1)$$

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\frac{1}{L}\sum_{n\neq m}^{L}\sqrt{\overline{|H_{nm}|^2}}$$

determines the nataure of eigenstates in the thermodynamic limit $L \to \infty$

$S \to \text{const}$	localized states
$S \sim L^{\alpha}, \quad \alpha > 0$	extended states
	critical states

Strong multifractality in the power-law ensemble

$b \ll 1$

General expression:

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\frac{1}{L}\sum_{n\neq m}^{L}\sqrt{\overline{|H_{nm}|^2}}$$

Power-law banded matrices:

$$\overline{|H_{nm}|^2} = \frac{1}{1+(|n-m|/b)^2}$$

$$\overline{I_q} = 1 - \sqrt{2\pi}\frac{\Gamma(q-1/2)}{\Gamma(q-1)} b \ln L$$

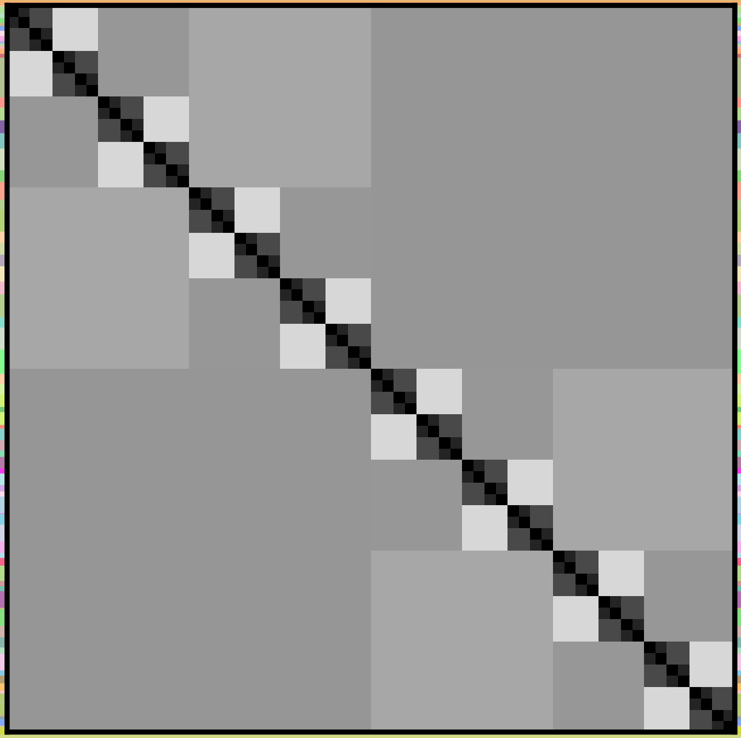
$$\overline{I_q} \propto L^{-d_q(q-1)} = 1 - d_q(q-1)\ln L + \frac{d_q^2(q-1)^2}{2}\ln^2 L + \dots$$

Fractal dimensions:

$$d_q = \sqrt{2\pi}\frac{\Gamma(q-1/2)}{\Gamma(q)} b$$

A. D. Mirlin and F. Evers, Phys. Rev. B **62**, 7920 (2000)

Structure of the ultrametric matrix



Metal-insulator transition in the ultrametric ensemble

$J/W \ll 1$

General expression:

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\frac{1}{L}\sum_{n\neq m}^{L}\frac{\sqrt{\overline{|H_{nm}|^2}}}{W}$$

Ultrametric ensemble:

$$\overline{|H_{nm}|^2} = \frac{J^2}{p^{d(n,m)-2}}$$

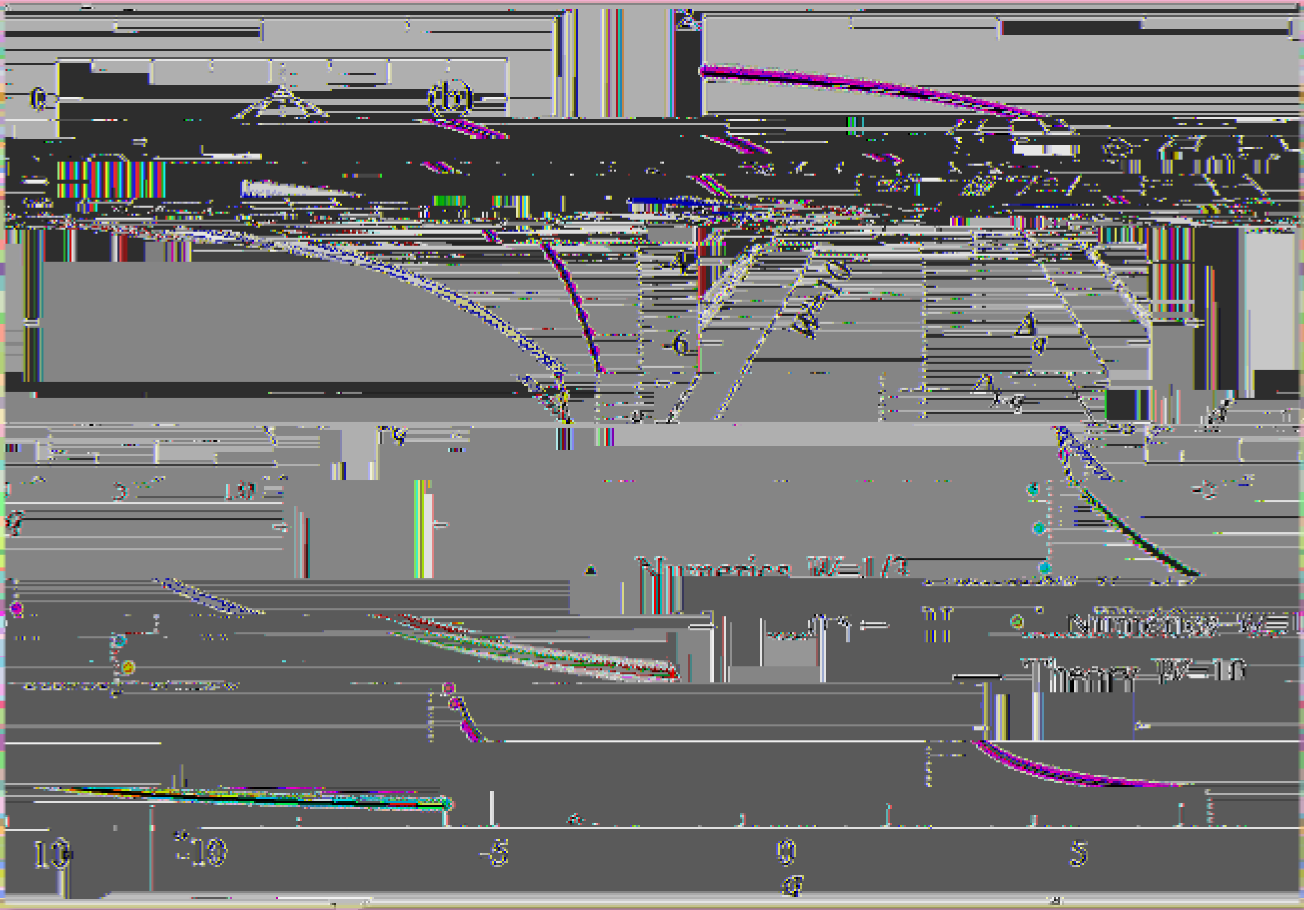
$$\overline{|H_{nm}|^2} = \frac{1}{(2/p)} \ldots \frac{(2/p)^{\log_2 L}-1}{(2/p)-1} \qquad S = \frac{1}{L}\sum_{n\neq m}^{L} \ldots$$

$p > 2$	$S \to \text{const}$	localized states
$p < 2$	$S \sim L^{\alpha}, \quad \alpha > 0$	extended states
$p = 2$	$S \sim \ln L$	critical states

I_q … $\sqrt{\pi}$ … $\Gamma(q-1/2)$ … $\sqrt{2}\ln 2$

Y. V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech., L12001 (2009)

Fractal dimensions in the ultrametric ensemble



Anomalous exponents:

$$\Delta_q \equiv (d_q - d)(q - 1)$$

Symmetry relation:

$$\Delta_q = \Delta_{1-q}$$

Y. V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech., L12001 (2009)

A. D. Mirlin et. al., Phys. Rev. Lett. **97**, 046803 (2007)

Universality of fractal dimensions

$$b = J/W \ll 1$$

Power-law banded matrices:

$$d_q = b\sqrt{2\pi}\,\frac{\Gamma(q-1/2)}{\Gamma(q)}$$

Ultrametric random matrices:

$$\frac{J}{W}\,\frac{\sqrt{\pi}}{\sqrt{2}\ln 2}\,\frac{\Gamma(q-1/2)}{\Gamma(q)}$$

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}}\,\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\,\frac{1}{L}\sum_{n\neq m}^{L}\frac{\sqrt{|H_{nm}|^2}}{W}$$

universality

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Fractal dimensions: beyond universality

$$d_q(b) = b f_1(q) + b^2 f_2(q) + \dots$$

$f_1(q)$ can be choosen the same for all models

Can we calculate $f_2(q)$?

Fractal dimension d_2 for power –law banded matrices

Supersymmetric virial expansion:

$$[\text{illegible}] = \frac{[\text{illegible}]}{\sqrt{2}} [\text{illegible}] \left[10 - [\text{illegible}] \right] [\text{illegible}]$$

where

$$[\text{illegible}] = \left(\frac{2}{\pi}\right)^3 \int_0^\pi \frac{[\text{illegible}]\, d\varphi_1\, d\varphi_2\, d\varphi_3}{(\cos\varphi_1 + \cos\varphi_2)(\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3)} \approx 0.79[\text{illegible}]$$

V. E. Kravtsov, A. Ossipov, O. M. Yevtushenko, E. Cuevas , Phys. Rev. B **82**, 161102(R) (2010)

Weak multifractality

$b = J/W \gg 1$

$$\Delta_q(b) = \frac{1}{b}F_1(q) + \frac{1}{b^2}F_2(q) + \ldots$$

How one can calculate $F_i(q)$?

Mapping onto the non-linear -model

Perturbative expansion in the regime $b \gg 1$

$$I_q(L) = \frac{q!}{L^{d(q-1)}}\left(1 + I_q^{(1)}(L) + I_q^{(2)}(L) + \ldots\right) \qquad I_q^{(1)} \propto \frac{1}{b}, \quad I_q^{(2)} \propto \frac{1}{b^2}$$

$$I_q^{(1)}(L) = -F_1(q)\frac{1}{b}\ln L + \ldots \qquad I_q^{(2)}(L) = \frac{F_1(q)^2}{2}\frac{1}{b^2}\ln^2 L - F_2(q)\frac{1}{b^2}\ln L$$

Non

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Spectral correlations

$$C(\omega) = \overline{|\psi_n(\mathbf{r})|^2 |\psi_m(\mathbf{r})|^2} \propto \omega^{\nu-1} \qquad \omega = E_n - E_m$$

$$I_2 = \sum_{\mathbf{r}} \overline{|\psi_n(\mathbf{r})|^4} \propto L^{-d_2}$$

Strong multifractality: $d_2 \to 0 \quad \Rightarrow \quad \nu - 1 \to -1$

Strong overlap of two infinitely sparse fractal wave functions!

J.T. Chalker and G.J.Daniell, Phys. Rev. Lett. **61**, 593 (1988)

Return probability

$$P(t) = \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} C(\omega)$$

$$C(\omega) \propto \omega^{\nu-1} \quad \Longleftrightarrow \quad P(t) \propto \frac{1}{(bt)^{\nu}}$$

$$\lim_{L^{d_2} bt/L \to \infty} R(t)\, P(t) = d_2 \propto \frac{1}{}$$

Strong multifractality $d_2, \nu \ll 1$:

$$\begin{cases} 1 - \nu \ln(bt) + \frac{\nu^2}{2} \ln^2(bt) + \dots & bt \ll L \end{cases}$$

$$\ln 4 + \pi I \quad + O(b^3) \quad \nu = d_2 = \frac{\pi b}{\sqrt{2}} + \frac{(\pi b)^2}{4\sqrt{2}} \left[10 - \frac{56}{3\sqrt{3}} - \dots \right]$$

V. E. Kravtsov, A. Ossipov, O. M. Yevtushenko, E. Cuevas , Phys. Rev. B **82**, 161102(R) (2010)