# Fluctuations and Extreme Values in Multifractal Patterns[1]

**Yan V Fyodorov**

**School of Mathematical Sciences**
Queen Mary Universi

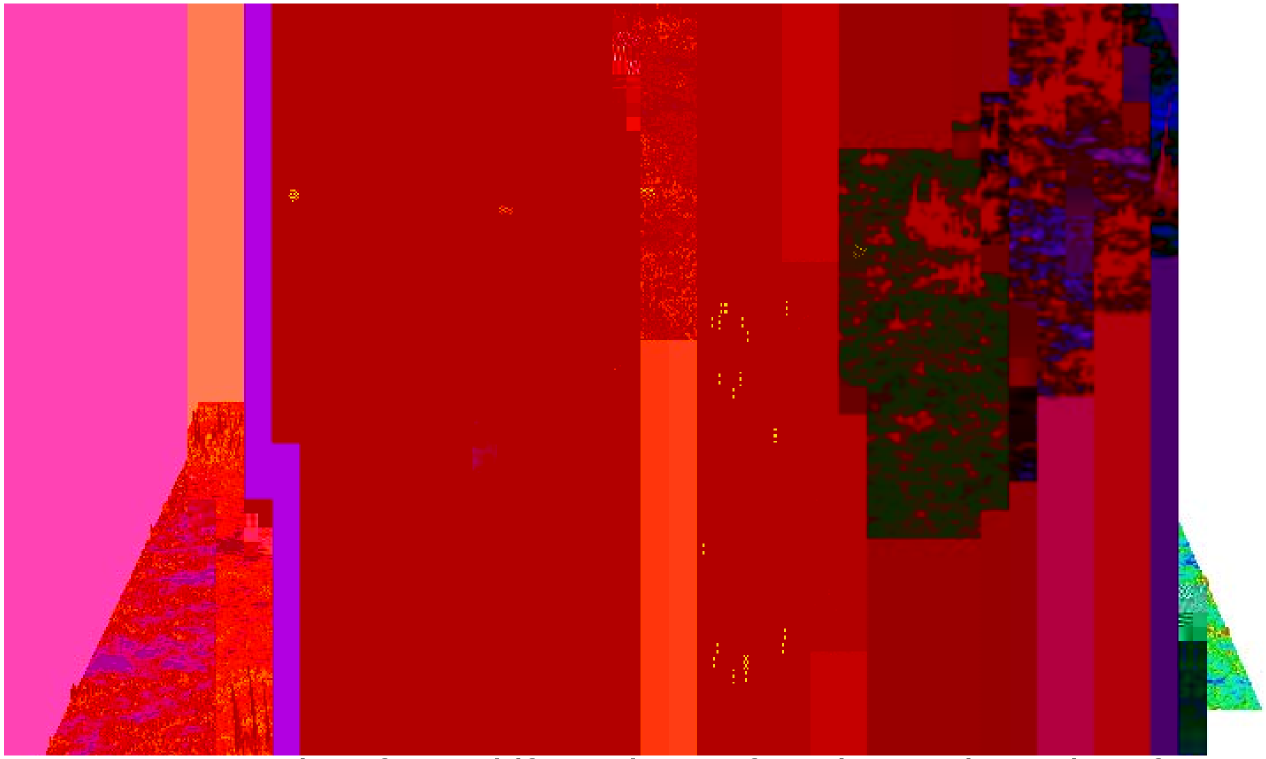
Project supported by the EPSRC grant EP/J002763/1

**VIII Brunel-Bielefeld Workshop on Random Matrix Theory**, 14th of December 2012

[1]Based on: **YVF**, **P Le Doussal** and **A Rosso** J Stat Phys: **149** (2012), 898-920
**YVF**, **G Hiary**, **J Keating** Phys. Rev. Lett. 108 , 170601 (2012) & **arXiv:1211.6063**

## Disorder-generated multifractals:

Disorder-generated multifractal patterns display high variability over a wide range of space or time scales, associated with huge fluctuations in intensity which can be visually detected. Another common feature is presence of certain long-ranged **powerlaw-type correlations** in data values.



Intensity of a multifractal wavefunction at the point of Integer Quantum Hall Effect.

Courtesy of F. Evers, A. Mirlin and A. Mildenberger.

## From disorder-generated multifractals to log-correlated fields:

$L$-gouslar

Disorder-generated multifractal patterns of intensities $h(\mathbf{r})$ are typically **self-similar**

$$\mathbb{E}\{h^q(\mathbf{r}_1)h^s(\mathbf{r}_2)\} \propto \frac{L}{a}^{y_{q;s}} \; |\mathbf{r}_1 - \mathbf{r}_2|$$

w 0 0 m 1 1 . 322 0 l S Q BT / f . 0 .

$h$

$r$

$a$

$\mathbf{r}$

## Ideal Gaussian periodic 1/f noise:

We will use a (regularized) model for ideal Gaussian periodic **1/f** noise defined as

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( v_n e^{int} + \overline{v}_n e^{-int} \right), \quad t \in [0, 2\pi)$$

where $v_n, \overline{v}_n$ are **complex standard Gaussian i.i.d.** with $\mathbb{E}\{v_n \overline{v}_n\} = 1$. It implies the formal covariance structure:

$$\mathbb{E}\{V(t_1)V(t_2)\} = -2\ln\left|2\sin\frac{t_1 - t_2}{2}\right|, \quad t_1 \neq t_2$$

**Regularization procedure (YVF & Bouchaud 2008):** subdivide the interval $[0, 2\pi)$ by a finite number $M$ of observation points $t_k = \frac{2\pi}{M}k$ where $k = 1, \ldots, M$, and replace the function $V(t), t \in [0, 2\pi)$ with a sequence of $M$ random mean-zero Gaussian variables $V_k$ correlated according to the $M \times M$ **covariance matrix** $C_{km} = \mathbb{E}\{V_k V_m\}$ such that the off-diagonal entries are given by

$$C_{k \neq m} = -2\ln|2\sin \frac{}{}$$

## Circular-logarithmic model (YF & Bouchaud 2008):

An example of the $1/f$ signal sequence generated for $M = 4096$ according to the above prescription is given in the figure.

The upper line marks the typical value of the **extreme value threshold** $V_m = 2 \ln M - \frac{3}{2} \ln \ln M$. The lower line is the level $\frac{1}{\sqrt{2}} V_m$ and blue dots mark points supporting $V_i > \frac{1}{\sqrt{2}} V_m$.

**Questions we would like to answer:** How many points are typically above a given level of the noise? How strongly does this number fluctuate for $M \to \infty$ from one realization to the other? How to understand the typical position $V_m$ and statistics of the **extreme values** (maxima or minima), etc. And, after all, what parts of the answers are **universal** and what is the universality class?

## Statistics of the counting function $N_M(x)$ and threshold of extreme values:

By relating moments of the counting function $N_M(x) = \int_x^\infty \rho_M(y)\,dy$ for log-correlated $1/f$ **noise** to **Selberg integrals** we are able to show that the probability density for the (scaled) counting function $n = N_M(x)/N_t(x)$ is given by:

$$P_x(n) = \frac{4}{x^2} e^{-n^{-\frac{4}{x^2}}} n^{-1+\frac{4}{x^2}}; \quad n \le n_c(x); \quad 0 < x < 2.$$

with $n_c \to \infty$ for $M \to \infty$ and the **characteristic scale** $N_t(x)$ given by

$$N_t(x) = \frac{M^{1-x^2/4}}{x\sqrt{\pi \ln M}} \frac{1}{(1-x^2/4)} = \mathbb{E}\{N_M(x)\} \frac{1}{(1-x^2/4)}$$

**Note:** For $x \to 2$ the **typical**

## **From** 1=*f* **noise to Riemann** $(1{=}2 + it)$**:**

One can argue that **log-mod** of the Riemann zeta-function $(1{=}2 + it)$ **locally** resembles a (non-periodic) version of the **1/f noise**. One can exploit this fact to predict statistics of **moments** and **high values** of the Riemann zeta along the critical line using the previously exposed theory (**YVF, Hiary, Keating** 2012).

## **Our approach to statistics of** $(1{=}2 + it)$**:**

We expect a **single** unitary matrix of size $N_T = \log(T{=}2\ )\quad 1$ to model the Riemann zeta $(1{=}2 + it)$, ) statistically, over a rangee

## Our predictions for $(1{=}2 + it)$ and CUE characteristic polynomials:

For the **maximum value**: $_{max}(T) = \max_{T \quad t \quad T+2} \; j \; (1{=}2 + it)j)$ we expect

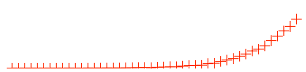
$$\log \; _{max}(T) \quad \log N_T \quad \tfrac{c}{2} \log \log N_T + \quad + \text{ [rand. noise}$$

## Our predictions for $(1{=}2 + it)$ and CUE characteristic polynomials:

**We further expect**

$$\log \ _{max}(T) \quad \log N_T \quad \tfrac{3}{4}\log\log N_T \quad \tfrac{1}{2}x; \quad N_T = \log(T{=}2 \ )$$

**where $x$ is distributed with a probability density behaving in the tail as** $(x \ ! \ \ 1) \quad jxj\, e^{x}$**.**



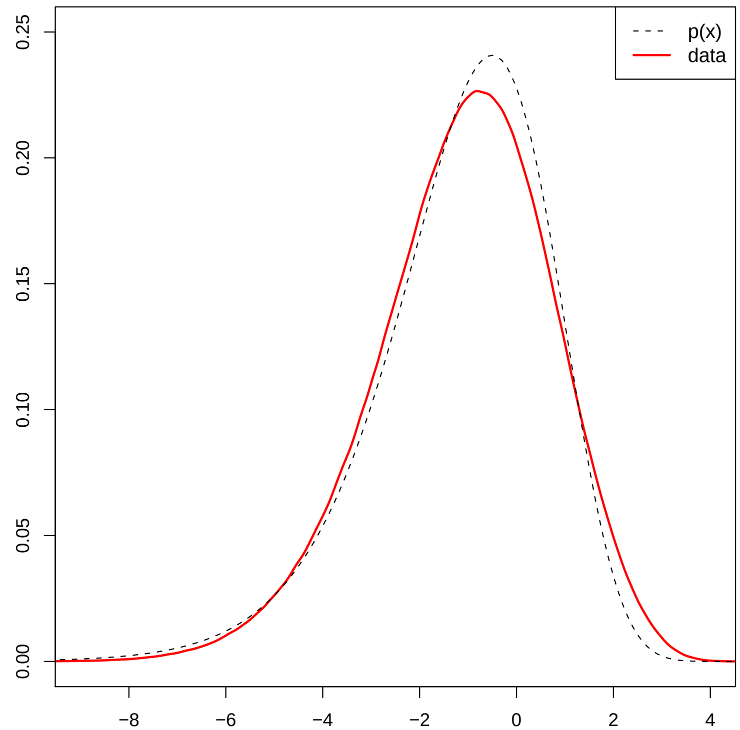



Figure 1: **Statistics of maxima for CUE polynomials (left: $N = 50; 10^6$ samples ) and $j \ (1{=}2 + it)j$ (right: $N_T = 65; \ 10^5$ samples ) compared to periodic $1{=}f$ noise prediction $p(x) = 2e^{x} K_0(2e^{x{=}2})$.**

## Threshold of extreme values for self-similar multifractal fields:

The value $c = \frac{3}{2}$ is a universal feature of systems with **logarithmic** correlations.

Apart from $1/f$ noise and its incarnations (characteristic polynomials of random matrices & zeta-function along the critical line) the new universality class is believed to include the $2D$ Gaussian free field, branching random walks & polymers on disordered trees, some models in turbulence and financial mathematics and, with due modifications the **disorder-generated multifractals**.

Namely, consider a multifractal random **probability measure** $p_i \sim M^{-\alpha_i}; i = 1, \ldots, M$ such that $\sum_{i=1}^{M} p_i = 1$ characterized by a general non-parabolic **singularity spectrum** $f(\alpha)$ with the left endpoint at $\alpha = \alpha_- > 0$. Then very similar consideration based on insights from **Mirlin** & **Evers** 2000 suggests that the **extreme value threshold** should be given by $p_m = M^{-\alpha_m}$, where $\alpha_m$

$$\alpha_m = \alpha_- + \frac{3}{2}\frac{1}{f'(\alpha_-)}\frac{\ln \ln M}{\ln M} \quad ) \quad \ln p_m \approx -\alpha_- \ln M + \frac{3}{2}\frac{1}{f'(\alpha_-)} \ln \ln M$$

## Threshold of extreme values for self-similar multifractal fields:

**Work in progress:** testing such a prediction for multifractal eigenvectors of a $N$ $N$ random matrix ensemble introduced by E. Bogomolny & O. Giraud, *Phys. Rev. Lett.* **106** 044101 (2011) based on **Rujsenaars-Schneider** model of $N$ interacting particles. Preliminary numerics is supportive of the theory.

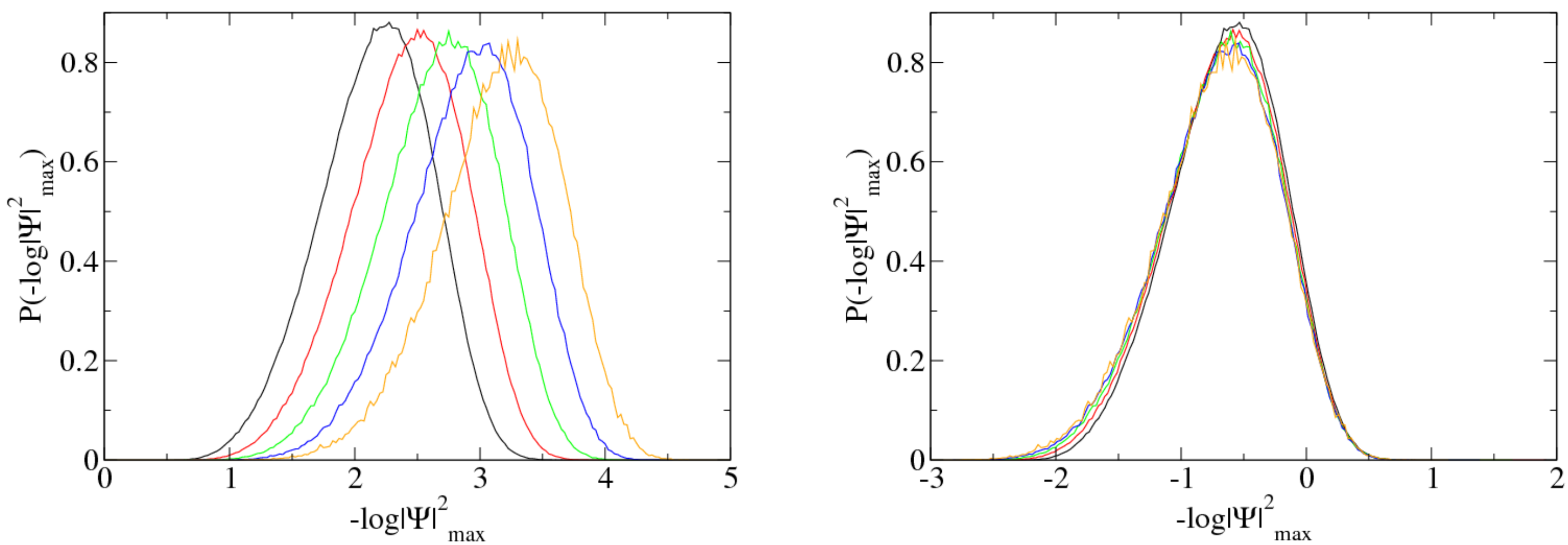



Figure 2: **Statistics of maxima for eigenvectors of RS model for sample sizes** $M = 2^n$ **with** $n = 8; : : : ; 12$**.**
**left:** **raw data** **right:** **each curve is shifted by** $\ln M + \frac{3}{2}\frac{1}{f'(\quad)}\ln\ln M$ **; data by Olivier Giraud**