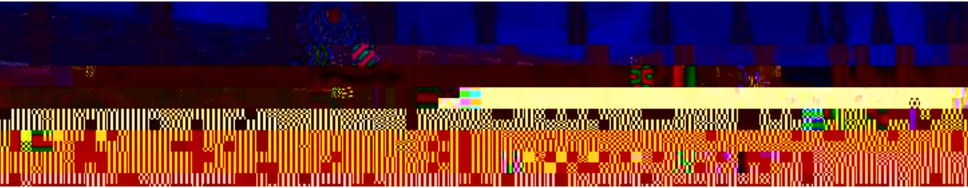

# Outline

# Model : random Gaussian landscape

## Hamiltonian (energy)

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# Glass transition for single particle at zero temperature

$$\mathcal{H}(\mathbf{x}) = \frac{\mu}{2}\sum_{k=1}^{N} x_k^2 + V(x_1, \ldots, x_N)$$

: random energy surface ($N$ dimensions)

Replica trick analysis (temperature $T$) $\Rightarrow$ critical value $\boxed{\mu_c = \overline{f''(0)}}$.

[Mézard, Parisi (1991)] [Fyodorov, Sommers (2007)]

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**[Mézard, Parisi (1991)] [Fyodorov, Sommers (2007)]**

**At $T = 0$ :**

$\mu < \mu_c$

# Glass transition for single particle at zero temperature

X

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# Outline

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hN

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$$\rho_m(x) = \det \partial^2_{i,j}\mathcal{H}) \theta \partial^2_{i,j}\mathcal{H}) \prod_{k=1}^{N} \delta(\partial_k \mathcal{H}) \quad V$$

minima

stationary points

**Heaviside** : $\theta(A) = $
1 if $A$ positive definite matrix
0 otherwise

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# Link with RMT

$$\rho_m(x) = \left\langle \det\left[\partial^2_{i,j}\mathcal{H}\right] \theta\left[\partial^2_{i,j}\mathcal{H}\right] \prod_{k=1}^{N} \delta\left(\partial_k \mathcal{H}\right) \right\rangle_V$$

**Translational invariant**

$$\rho_m(\mathbf{x}) = \left\langle \left| \det \left[ \partial^2_{i,j}\mathcal{H} \right] \right| \theta \left[ \partial^2_{i,j}\mathcal{H} \right] \prod_{k=1}^{N} \delta\left(\partial_k \mathcal{H}\right) \right\rangle_V$$

**Translational invariant covariance structure of $V$**
$\Rightarrow \langle \partial_k V \, \partial_i \partial_j V \rangle = 0$ (at same $\mathbf{x}$) and $\langle \partial_j V \partial_k V \rangle = -\delta_{j,k} f'(0) \equiv \delta_{j,k}\sigma^2$.
**Thus**

$$\boxed{\rho_m(\mathbf{x}) = \frac{1}{(\sqrt{2\pi\sigma^2})^N} e^{-\frac{\mu^2 \mathbf{x}^2}{2\sigma^2}} \left\langle \left|\det\left(\mu \, \mathrm{Id} - M\right)\right| \, \theta\left(\mu \, \mathrm{Id} - M\right) \right\rangle_M}$$

Hessian $= \partial^2_{i,j}\mathcal{H} = \mu \, \mathrm{Id} - M$ with $\boxed{M_{i,j} = -\partial_i \partial_j V}$

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$M$ = **Gaussian random matrix**, law independent of $\mathbf{x}$ :

$$\begin{aligned} \langle M_{i,j} \rangle &= 0 \\ \langle M_{k,l} M_{i,j} \rangle &= \frac{\mu_c^2}{N} (\delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k} + \delta_{i,j}\delta_{k,l}) \end{aligned}$$

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$$\boxed{\langle \mathcal{N}_m \rangle = \frac{1}{\mu^N} \, \langle |\det(\mu \, \mathrm{Id} - M)| \; \theta(\mu \, \mathrm{Id} - M) \rangle_M}$$

# Link with standard Gaussian ensembles of matrices

$$\langle \mathcal{N}_m \rangle = \frac{1}{\mu^N} \langle |\det(\mu - M)| \; \theta(\mu - M) \rangle_M \quad \text{with} \quad \mathcal{P}(M) \propto e^{-\frac{N}{4\mu_c^2}\left[\mathrm{tr} M^2 - \frac{(\mathrm{tr} M)^2}{N+2}\right]}$$

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**Introduce additional Gaussian integration :**

$$\langle \mathcal{N}_m \rangle = \frac{1}{\mu^N} \int_{-\infty}^{+\infty} dt \sqrt{\frac{N}{2\pi}} \, e^{-N\frac{t^2}{2}} K_N(z_t) \quad \text{with} \quad z_t = \mu + \mu_c t$$

$$\boxed{K_N(z) = \left\langle \, |\det(z - M_0)| \; \theta(z - M_0) \right\rangle_{M_0}}$$

**Gaussian Orthogonal Ensemble (GOE) :**

$$P(M_0) = C_N \; \exp$$

# From the matrix to the eigenvalues

$$\boxed{K_N(z) = \left\langle \, |\det(z - M_0)| \;\, \theta(z - M_0) \, \right\rangle_{M_0}} \quad \textbf{with} \quad P(M_0) = C_N \; e^{-\frac{N}{4\mu_c^2} \operatorname{tr} M_0^2}$$

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- $O(N)$ invariance of **GOE** measure

  $\Rightarrow$ introduce the **eigenvalues** $\lambda_i$ of $M_0$

$$\boxed{K_N(z) = \frac{1}{z_N} \int_{-\infty}^{z} d\lambda_1 \ldots \int_{-\infty}^{z} d\lambda_N \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} |z - \lambda_i| \, e^{-\frac{N}{4\mu_c^2}\lambda_i^2}}$$

$\prod_{i<j} |\lambda_i - \lambda_j|$ : Vandermonde determinant (Jacobian).

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# Relation with maximal eigenvalue of GOE

- **We want to compute :**

$$\tilde{\kappa}_N(y) = \int^{y} d\lambda_1 \ldots \int^{y} d\lambda_N \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} (y - \lambda_i)\, e^{-\frac{\lambda_i^2}{2}}$$

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- **For standard GOE, ie** $P(M_{GOE}) \propto e^{-\frac{1}{2}\mathrm{tr} M_{GOE}^2}$ **:**

$$P(\lambda_1, \ldots, \lambda_N) = D_N \prod_{i<j} |\lambda_i - \lambda_j| \prod_i e^{-\frac{\lambda_i^2}{2}}$$

# Relation with maximal eigenvalue of GOE

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# Number of Minima and GOE eigenvalues

$$\mathcal{H} = \frac{\mu}{2} \sum_{k=1}^{N} x_k^2 + V$$

**Finally ,we get :**

$$\langle \mathcal{N}_m \rangle = \left( \frac{\mu_c}{\mu} \right)^N B_N \; I_N(\mu/\mu_c)$$

**with** $I_N(\mu/\mu_c) = \int_{-\infty}^{+\infty} dy \; e^{\frac{y^2}{2} - \frac{N}{2}\left( y\sqrt{\frac{2}{N}} - \frac{\mu}{\mu_c} \right)^2} \frac{d}{dy} [\mathbb{P}_{N+1}(\lambda_{max} \leq y)]$

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$$\mathcal{H} = \frac{\mu}{2}\sum_{k=1}^{N} x_k^2 + V$$

$$\mu_c = \overline{f''(0)}$$

($V$ : random part)

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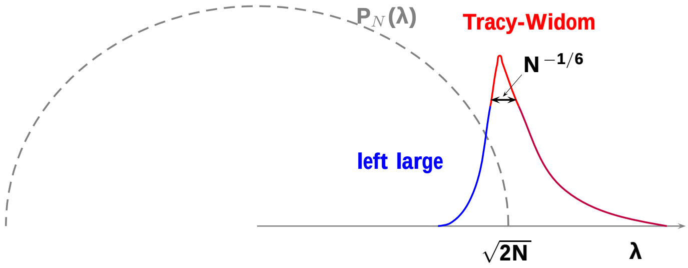
# Outline

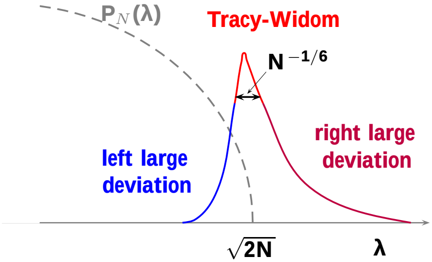
# Maximal eigenvalue of GOE : large deviations as $N \to \infty$



# Maximal eigenvalue of GOE : large deviations as $N \to \infty$

**Large deviations of pdf of $\lambda_{max}$ for GOE**
**[Dean, Majumdar (2006)]**
**[Majumdar, Vergassola (2009)]**

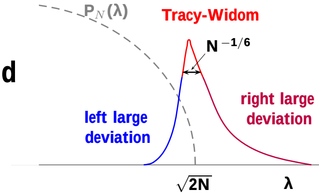
**d**



# Correspondance between $\lambda_{max}$ and our glass transition

$$\langle \mathcal{N}_m \rangle = \left(\frac{\mu_c}{\mu}\right)^N B_N \; dy \; e^{\frac{y^2}{2} - \frac{N}{2}\left(y\sqrt{\frac{2}{N}} - \frac{\mu}{\mu_c}\right)^2} \frac{d}{dy}\left[\mathbb{P}_{N+1}(\lambda_{max} \le y)\right]$$

**Use large deviations of $\lambda_{max}$ + saddle point method as $N \to \infty$**

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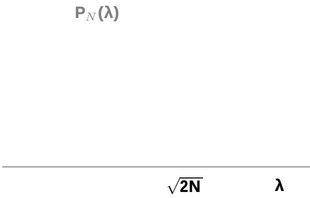

# Glassy phase (logarithmic equivalent for $N \to \infty$)

$$\langle \mathcal{N}_m \rangle = \frac{\mu_c}{\mu}^{N} B_N \quad dy\, e^{\frac{y^2}{2} - \frac{N}{2}\left(y\sqrt{\frac{2}{N}} - \frac{\mu}{\mu_c}\right)^2} \frac{d}{dy}\left[\mathbb{P}_{N+1}(\lambda_{max} \le y)\right]$$

**Use large deviations of $\lambda_{max}$ + saddle point method**

$$\frac{d}{dy}\left[\mathbb{P}_N(\lambda_{max} \le y)\right] \approx \begin{cases} e^{-N^2 \psi_-(s)} & \text{for } s < \sqrt{2} \\ e^{-N \psi_+(s)} & \text{for } s > \sqrt{2} \end{cases}$$

$\Rightarrow$ **recover the result of [Fyodorov, Williams (2007)] :**

| $\mu < \mu_c$ | $\mu > \mu_c$ |
|---|---|
| $\langle \mathcal{N}_m \rangle \approx e^{N \Sigma\left(\frac{\mu}{\mu_c}\right)}$ | $\langle \mathcal{N}_m \rangle \approx O(1)$ |
| $\Sigma(\ ) = -\ln(\ ) - \frac{m^2}{2} + 2 \quad - \frac{3}{2}$ | |
| **glassy phase : random part ($V$) dominates** | **harmonic part dominates** |

**critical point $\mu_c$ = $\overline{f''(0)}$**

# A step further (1) : more detailed large deviation results

**Large deviations of $\frac{d}{dy}[\mathbb{P}_N(\lambda_{max} \leq y)]$ for GOE : $y = s\sqrt{N}$, large**

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**Large deviations of $\frac{d}{dy}[\mathbb{P}_N(\lambda_{max} \leq y)]$ for GOE : $y = s\sqrt{N}$, large $N$**

- **Left tail : $y < \sqrt{2N}$, ie $s < \sqrt{2}$**

$$\frac{d}{dy}[\mathbb{P}_N(\lambda_{max} \leq y)] \sim e^{-N^2\psi_-(s)+N\Phi_1(s)-\phi_1 \ln N-\Phi_2(s)}$$

cf [Borot, Eynard, Majumdar, Nadal (2011)] : left large deviations of $\lambda_{max}$ for G$\beta$E (here $\beta = 1$) using loop equations.

- **Right tail : $y > \sqrt{2N}$, ie $s > \sqrt{2}$**

$$\frac{d}{dy}[\mathbb{P}_N(\lambda_{max} \leq y)] \sim \frac{e^{-N\psi_+(s)}}{2\sqrt{\pi}(-2+s^2)^{1/4}\sqrt{s+\sqrt{-2+s^2}}}$$

cf [Borot, Nadal (2012)] : right large deviations for G$\beta$E (loop equations).

# A step further (2) : exact equivalent for both phases

**Using previous results and saddle point method, get large $N$ equivalents :**

- **Phase where the harmonic potential dominates : $\mu > \mu_c$**

$$\boxed{\langle \mathcal{N}_{\mathbf{m}} \rangle \sim 1}$$

# A step further (2) : exact equivalent for both phases

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**Using previous results and saddle point method, get large $N$ equivalents :**

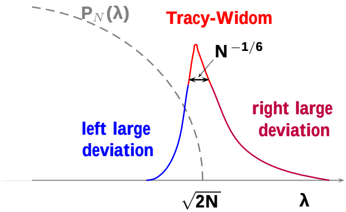
- 

# Outline

# Close to transition : Tracy-Widom (1)

**GOE, fluctuations of $\lambda_{max}$ close to mean value, ie $\lambda_{max} - \sqrt{2N} = O(N^{-1/6})$ :**

$$\mathbb{P}_N \quad \frac{\lambda_{max} - \sqrt{2N}}{N^{-\frac{1}{6}}/\sqrt{2}} \le x \quad \sim \mathcal{F}_1(x$$



# Close to transition : Tracy-Widom (2)

**Correspondance between GOE and our disordered system :**

| | $\lambda_{max}$ for GOE | Curvature (disordered system) |
|---|---|---|
| Critical point | $y_c = \sqrt{2N}$ | $\mu_c = \overline{f''(0)}$ |

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**Transition regime :**

$$\langle \mathcal{N}_m \rangle \sim \mathcal{N}(\delta) \qquad \text{for} \quad \frac{\mu}{\mu_c} = 1 + \delta N^{-1/3} + \ldots$$

# Phases in disordered system $\mathcal{H}(\mathbf{x}) = \frac{\mu}{2}$

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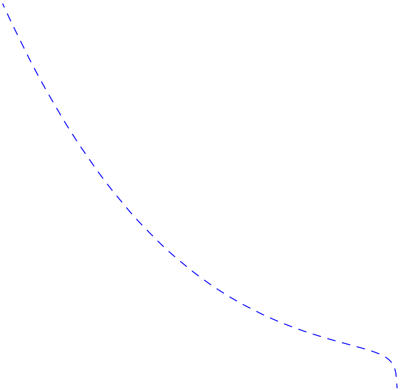

# Phases in disordered system $\mathcal{H}(\mathbf{x})$ =

# Matching between di erent regimes

**Intermediate regime :** $\langle\mathcal{N}$

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- **Matching with the right tail :** $\delta \to \infty$

**Right asymptotics of Tracy-Widom :** $1 - \mathcal{F}_1(x) \sim \dfrac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{4\sqrt{\pi}\, x^{3}}$

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- **Matching with the right tail** : $\delta \to \infty$

  Right asymptotics of Tracy-Widom : $1 - \mathcal{F}_1(x) \sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{4\sqrt{\pi}\, x^{\frac{3}{4}}}$ as $x \to \infty$

  Find $x^* \sim \delta^2$ and recover phase $\mu > \mu_c$ : $\boxed{\langle \mathcal{N}_m \rangle \sim 1}$.

- **Matching with the left tail** : $\delta \to -\infty$

  Left asymptotics of TW : $\mathcal{F}_1(x) \sim \tau_1 \frac{e^{-\frac{1}{24}|x|^3 - \frac{1}{3\sqrt{2}}|x|^{\frac{3}{2}}}}{|x|^{\frac{1}{16}}}$ as $x \to -\infty$

  Find $x^* \sim -2\sqrt{-2\delta}$ and

$$\boxed{\langle \mathcal{N}_m \rangle \sim 2\, 2^{\frac{21}{32}} \sqrt{\pi} |\delta|^{\frac{23}{32}}\, \tau_1\, e^{\frac{|\delta|^3}{3} + \frac{4\sqrt{2}}{3}|\delta|^{\frac{3}{2}} - \frac{2^{7/4}}{3}|\delta|^{\frac{3}{4}} + \frac{1}{4}}}$$

# Matching between different regimes

**Intermediate regime :** $\boxed{\langle \mathcal{N}_{\mathbf{m}} \rangle \sim \mathcal{N}(\delta)}$ **for** $\frac{\mu}{\mu_c} = 1 + \delta N^{-1/3} + \ldots$

- **Matching with the right tail** : $\delta \to \infty$

**Right asymptotics of Tracy-Widom :** $1 - \mathcal{F}_1(x) \sim \quad e$

# Conclusion

**Study of the**

Thank you !

**Reference :**

**Yan V. Fyodorov and Céline Nadal,**

**“Critical Behavior of the Number of Minima of a Random Landscape**