Random Fermionic Systems

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Background

- First introduced to study magnetic properties of matter
- Toy model for quantum information { study of entanglement
- Random matrix aspect

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Three papers that inspired this work:

- Lieb-Schultz-Mattis \Two soluble models of an Antiferromagnetic chain"
- Doctoral thesis of Huw Wells supervised by Jon K4.976 cm 1d bychain"

Our object of study: the Hamiltonian

- Self-adjoint operator acting on $\mathbb{C}^{2^n}$
- $$H = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij}(c_i^\dagger c_j - c_i c_j^\dagger) + B_{ij}(c_i c_j - c_i^\dagger c_j^\dagger)$$

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  with $A_{ij} = A_{ji}, \quad B_{ij} = -B_{ij}$, i.e. $A = A^t$ and $B = -B^t$.
- c_j's are fermionic i.e. $\{c_i, c_j\} = 0, \{c_i, c_j^{\dagger}\} = \delta_{ij}$,

We take A_{ij}, B_{ij} iid real. Our conclusions:

- Ground state energy gap $O(1/n)$ with explicit formula if Gaussian entries
- DOS – Gaussian universally, also for A, B band
- No repulsion – numerics

Universality

- Gaussian DOS vastly universal
- Subset sums: given a set $\{\lambda_1, \ldots, \lambda_n\}$ and $S_j \subseteq \{1, \ldots, n\}$, eigenvalues of H are closely related to $\sum_{k \in S_j} \lambda_k$.
- A lot of information
  - Gaussian DOS
  - Groundstate energy gap

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- Relation to sums of weighted binomial random variables – can take Fourier transform explicitly!

Fermionic systems: how they arise?

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- nearest neighbor interaction { the XY model

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- n sites with spins that are linear combinations of ^{x} and ^{y} (no ^{z})
- nearest neighbor interaction { the XY model
- the corresponding Hamiltonian is

$$\sum_{k=1}^{n} \sum_{a2fx;yg} \sum_{b2fx;yg} \quad _{k;a;b} \quad ^{a}_{k} \quad ^{b}_{k+1}$$

- Here $^{(a)}_{j} = I_2^{(j \quad 1)} \quad ^{(a)} \quad I_2^{(n \quad j)}$

Jordan-Wigner transformation

- Maps a spin chain to a quadratic form in fermionic operators: allows for an exact solution
- In reverse: model a system of interacting fermions on a quantum computer

Jordan-Wigner details

- Raising and lowering operators $a_i^y = \ {}_i^x + i \ {}_i^y$ and $a_i = \ {}_i^x \quad i \ {}_i^y$
- Can recover Pauli spin operators by ${}_j^x = (a_j^y + a_j)=2$, ${}_j^y = (a_j^y \quad a_j)=2$, ${}_j^z = (a_j^y a_j \quad 1=2)$

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- Not fermionic
  - Partly fermionic: $f a_j ; a_j^y g = 1; a_j^2 = (a_j^y)^2 = 0$
  - Partly bosonic: $[a_j^y ; a_k^y] = [a_j^y ; a_k^y] = [a_j ; a_k] = 0$
- For fermionic let

$$c_j = \exp \left(i \sum_{k=1}^{j-1} a_k^y a_k \right) a_j$$

$$c_j^y = a_j^y \exp \left(i \sum_{k=1}^{j-1} a_k^y a_k \right) :$$

c_j's and c_j^y's are fermionic: $f c_j ; c_k^y g = \ {}_{kj} ; f c_j ; c_k g = f c_j^y ; c_k^y g = 0$

Lieb-Schultz-Mattis Antiferromagnetic Chain '61

- $H = \mathrm{P}_j (1 + \quad)\;{}^{x}_{j}\;{}^{x}_{j+1} + (1 \qquad)\;{}^{y}_{j}\;{}^{y}_{j+1}$
- Hamiltonian is a quadratic form in Fermi operators and can be explicitly diagonialized

Lieb-Schultz-Mattis

If $H = (\underline{c}^{\dagger}\ \underline{c})\ M \begin{pmatrix} \underline{c} \\ \underline{c}^{\dagger} \end{pmatrix}$; with $M = \frac{1}{2}\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ for XY model as before

$$A = \begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & 1 \\ 1 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & 1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & 1 \\ 1 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & 1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & 1 & 0 \end{pmatrix}$$

and A, B can be explicitly diagonalized.

In the '61 paper,

- Complete set of eigenstates
- General expression for the order between any two spins involving a Green's function
- Short, intermediate, and long range order for various situations

Bipartite Entanglement

Setup: XY and XX models with a constant transversal magnetic eld
Study: Entropy E_p of entanglement between subsystems

- Vidal et al. computed E_p numerically
- Jin and Korepin compute E_p for XX model using the Fisher-Hartwig conjecture, which gives the leading order asymptotics of determinants of certain Toeplitz matrices
- Keating and Mezzadri study asymptotics of entanglement of formation of ground state using RMT methods

Wells PhD thesis

Hamiltonians of the form

$$H_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{a=1}^{3} \sum_{b=1}^{3} {}_{a;b;j} \; {}_{j}^{(a)} \; {}_{j+1}^{(b)} \qquad (1)$$

for any ${}_{a;b;j} \in \mathbb{R}$ random Gaussian (some universality possible)

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for any $_{a;b;j}$ 2 R random Gaussian (some universality possible)

Remarks:

Wells Numerics in the XY case

For a Hamiltonian of the form

$$H_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{a=1}^{2} \sum_{b=1}^{2} \alpha_{a,b,j}\, \sigma_j^{(a)} \sigma_{j+1}^{(b)} \tag{2}$$

- Eigenvalue repulsion in the full model and lack of repulsion in the random XY model
- Convergence to a Gaussian in the random XY model
- Numerical estimate of the error in the random XY model is on the order of $1/n$ where n is the number of cubits

Extension by Erdős and Schröder

- Arbitrary graphs with maximal degree total number of edges
  - Gaussian DoS
- p-uniform hypergraphs
  - Correspond to p-spin glass Hamiltonians acting on n distinguishable spin-1/2 particles
  - At $p = n^{1/2}$, phase transition between the normal and the semicircle
  - **quantum-classical** transition

Summary

Known:

- DoS, spectral gap in (deterministic) XY model
- DoS in a random neighbor-to-neighbor Hamiltonian with XYZ

Numerics:

- DoS in a random XY model
- Rate of convergence in the random XY model
- Lack of repulsion

We establish:

- DoS in general bilinear forms of fermionic operators
- spectral gap in special cases

Diagonalizing M

- Eigenvalue equation: $\frac{1}{2}\begin{matrix} A & B \\ B & A \end{matrix} \quad \begin{matrix} 1 \\ 2 \end{matrix} = \quad \begin{matrix} 1 \\ 2 \end{matrix} :$
- Equivalent to: $\left(\begin{matrix} A \ _1 \quad B \ _2 = 2 \quad _1; \\ B \ _1 \quad A \ _2 = 2 \quad _2: \end{matrix} \right.$
- If $_1 = \ _1 \quad _2$ and $_2 = \ _1 + \ _2$, then $\left(\begin{matrix} (A + B) \ _1 = 2 \quad _2; \\ (A \quad B) \ _2 = 2 \quad _1: \end{matrix} \right.$

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- Note that $(A \quad B)^T = (A + B)$ and hence we get

$$\frac{1}{4}(A + B)^T (A + B) \ _1 = \ ^2 \ _1:$$

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- Note that $(A \quad B)^T = (A+B)$ and hence we get

$$\frac{1}{4}(A+B)^T(A+B)\ {}_1 = {}^2\ {}_1.$$

$$2 \quad (M) \quad (\ ) \quad p\overline{\ \ {}^2} \text{ is singular value of } \frac{A+B}{2}.$$

Need Hermiticity to get new Fermi operators

- Let U be the orthogonal matrix that diagonalizes M.
- Then U is a linear canonical transformation in the sense that

$$U = \begin{matrix} G & K \\ G^T & K^T \end{matrix} \qquad \begin{cases} GG^T + KK^T = I_n \\ GK^T + KG^T = 0_n; \end{cases} \tag{3}$$

and

$$UMU^T = \frac{1}{2} \begin{matrix} & 0 \\ 0 & \end{matrix} \quad ;$$

with $= \mathrm{diag}({}_1; \ldots; {}_n)$, ${}_i \quad 0$.

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with $\;\; = \mathrm{diag}(\;_1; \ldots ; \;_n)$, $\;_i \quad 0$.

- Let $\;_k; \;_k^y$ operators de ned by

–

Diagonalizing H: Fermi basis

- $_j$ acts as a lowering operator for $^y_j \; _j$ i.e. if $^y_j \; _j j \quad i = j \quad i$ then $^y_j \; _j \; _j j \quad i = 0$
- ^y_j acts as a raising operator for $^y_j \; _j$

Diagonalizing H: Fermi basis

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- $^\dagger_j$ acts as a raising operator for $^\dagger_j\ _j$
- $^\dagger_j\ _j$'s commute so there exists a state $|\ \rangle$ which is a simultaneous eigenstate

Diagonalizing H: Fermi basis

- ${}_j$ acts as a lowering operator for ${}^y_j\ {}_j$ i.e. if ${}^y_j\ {}_j j\ \ i = j\ \ i$ then ${}^y_j\ {}_j\ {}_j j\ \ i = 0$
- ${}^y_j$ acts as a raising operator for ${}^y_j\ {}_j$
- ${}^y_j\ {}_j$'s commute so there exists a state $j\ \ i$ which is a simultaneous eigenstate
- By raising and lowering the state $j\ \ i$ in all possible combinations, can construct a set of 2^n orthonormal states which are simultaneous eigenstates of the ${}^y_j\ {}_j$

Diagonalizing H : subset sums

The spectrum of H is characterized as follows:

$$\boxed{x \in \ (H) \quad (\ ) \quad \exists S \quad \{1,\dots,n\} \text{ such that } x = c + \sum_{k\in S} {}_k .} \qquad (4)$$

where $c = \ \frac{1}{2}\sum_{k=1}^{n} {}_k$

Ground state energy gap: important physical quantity, re ects how sensitive is the system to perturbations

Theorem 1

For A, B

Ground state energy gap: important physical quantity, re ects how sensitive is the system to perturbations

Theorem 1

For A, B with iid Gaussian entries up to symmetry, the rescaled energy gap $\sqrt{2n=}$ *converges in distribution to a random variable whose probability density function is*

$$f(x) = (1 + x)e^{\ \frac{x^2}{2}\ \ x}:$$

- $x_{2^n} = \sum_{j=1}^{n}\ _j$ and $x_{2^n\ 1} = \sum_{j=2}^{n}\ _j$ yielding that

$$:= x_{2^n}\quad x_{2^n\ 1} = \quad _1$$

- Recall that $\ _j$ are singular values of $A + B$
- Result for smallest eigenvalue value of Wishhart matrices by Edelman
- Note that   is very large compared to mean spacing (O(1/n) instead of $2^{\ n}$)

The relation with iid Bernoullis

Let x_j be the eigenvalues of H. Then

$$x_j = \frac{1}{2}\sum_{k\in S_j}\epsilon_k - \frac{1}{2}\sum_{k\in S_j^c}\epsilon_k$$

for some $S_j \subset \{1,\dots,n\}$.

Then

$$d\mu_n = \frac{1}{2^n}\sum_{j=1}^{2^n}\delta_{x_j} = \text{prob. meas. of } \sum_{j=1}^{n}\epsilon_j(B_j - 1/2)$$

where B_j are n independent Bernoulli random variables.

The relation with iid Bernoullis

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Then

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Details

1. Lindenberg condition states:
   - variances $_k$ are nite
   - $s_n^2 = \sum_{k=1}^{n} {}_k^2$
   - $\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathrm{E}\, (X_k)^2 \; \mathbf{1}_{\{|X_k| > \varepsilon s_n\}} = 0$
   - yields convergence to a Normal distribution with variance s_n for sequences of $_j$ so that the maximum $< \sqrt[4]{n}$
   - will show that the condition on the max is satis ed with P $\to$ 1 as $n \to \infty$
   - a Berry-Esseen estimate yields an error of $1/\sqrt{n}$

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   - will show that the condition on the max is satis ed with P $!$ 1 as $n\ !\ 1$
   - a Berry-Esseen estimate yields an error of $1{=}\sqrt{n}$
2. For the computation of the Fourier transform :
   1. Fourier transform of $\frac{1}{\sqrt{n}}\ _j(B_j\ \ 1{=}2)$ is cos $\frac{t\ j}{2\sqrt{n}}$
   2. Fourier transform of the DoS is then Q_j cos $\frac{t\ j}{2\sqrt{n}}$

Random Matrix Theory

Have to show that $_n$ $\sqrt{n}$ when ^{2} of matrix entries is $1=N$

Our Numerics

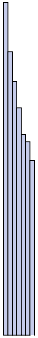

Figure: Spacing distribution for the unfolded spectrum.

Our Numerics

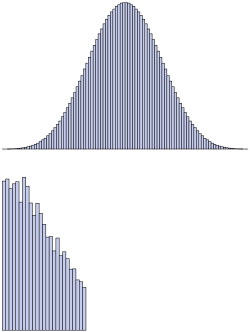

Figure: Density of states and ground state energy gap distribution for Gaussian quadratic form of Fermi operator. Here $n = 16$ (for a sample size of about 50).

Future study

Further questions we want to examine:

- Rate of convergence can probably be improved.
- The bottom eigenvalue of a band covariance matrix.
- In the bulk, the eigenvalues appear to form a Poisson process on the line.
- Speculation: relation to the Berry-Tabor conjecture. Generic integrable system) Poisson statistics

Thank you!